Notes on Cartesian Tensors and Mathematical Models of Elastic Solids and Viscous Fluids

Pascal Grange
Xi'an Jiaotong-Liverpool University, Department of Mathematical Sciences
*E-mail: pascal.grange@xjtlu.edu.cn

Last updated: Spring 2017

Preface

This document contains lecture notes and tutorials for the one-semester module Cartesian Tensors and Mathematical Models of Elastic Solids and Viscous Fluids, taught in the Applied Mathematics programme at Xi'an Jiaotong-Liverpool University (XJTLU) between 2014 and 2017. The format of lecture notes and tutorials has been kept and roughly reflects the pace of weekly delivery of the module. The organisation into chapters aims to guide the reader through the main themes. General concepts of continuum mechanics are introduced in the first four chapters, Chapter 6 and Chapter 5 are independent from each other and can be studied in any order.

Each tutorial sheet starts with Exercise Zero, which invites the reader to edit the present text and to report typos, errors or expository issues of any kind by email. These notes are therefore by nature a work in progress, but they benefitted from the answers given to Exercise Zero by several cohorts of XJTLU students. Substantial contributions by Zuqian Huang, Jin Yan, Ran Bi, Zhiwei Cheng, Qier Yu, Henger Li, Chang Liu and Chenxia Gu are gratefully acknowledged.

Contents

Chapter 1

Cartesian coordinates, vectors, tensors

Lecture 1 : from linear algebra to tensors

Keywords. Linear algebra, vectors, matrices, scalar product, change of orthonormal basis, transformation matrix, tensor, isotropic tensors.

1.1 Tensors and continuous media: motivations and history

Systems of coordinates are mathematical structures used to describe the properties of physical systems in a quantitative way. They depend on the choice of the person who designs the model (for example, a certain choice of axes can make computations easier). However, the physical results should not depend on the choice of coordinates.

• **This independence induces transformation laws on certain quantities.** If one changes the system of coordinates, the quantities we write change as well, but they must do so in a precise way in order for the physical reality to stay the same. Tensors are quantities defined by the way they transform when the system of coordinates changes. The word *Cartesian*[1] refers to the fact that one considers systems of coordinates in which the three axes form an orthogonal basis.

• **Continuous media: solids and viscous fluids.** We are interested in scales at which the (discrete, atomic, molecular[2]) microscopic structure of matter is not apparent: distances are large in scale of the size of molecules, and we will treat solids and fluids as continuous media on which we will put coordinates just as if we were on a geometric space. In some cases, such as concrete and most rocks (think of granit and marble), heterogeneities are detectable by the naked eye but quantitative modeling by a continuum proves satisfactory. The mathematical framework of continuous media was developed before atomic and molecular physics (for example the Euler equation for the mechanics of fluids was written in 1755[3]). At macroscopic scales it is still valid, because the huge numbers of microscopic constituents have interactions that result in collective behaviours that can be described at larger scales by a relatively small number of parameters (such as elasticity coefficients and density). These parameters are to be determined

[1] from the name of René Descartes (1596-1650). Under his influence, numbers started being used to describe geometric objects. His name is also attached to the laws of geometric optics.

[2] Let us mention that tensors are used for other purposes in atomic and subatomic physics, to address symmetry properties of particles.

[3] Leonhard Euler (1707-1783) founded the discipline of fluid mechanics, and contributed in a decisive way to all branches of mathematics.

experimentally, but this course will introduce the mathematical framework that can be described for all continuous media (fluids and solids).

1.2 Linear algebra reminders, notations and conventions

Unless otherwise stated, we will assume in this course that the ambient space is the three-dimensional space $\mathbf{R}^3$ (but tensors can be defined in any finite dimension, using the reasoning ofthe present document). This section recalls a few notions of linear algebra and introduces notations.

Let $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal of $\mathbf{R}^3$. Let O be the origin of a coordinate system. Let M be a point in $\mathbf{R}^3$. It can be described by its coordinates (x_1, x_2, x_3), the three numbers such that the vector $\vec{OM}$ is written as follows:

$$\overrightarrow{OM} = x_1\vec{e_1} + x_2\vec{e_2} + x_3\vec{e_3}. \tag{1.1}$$

1.2.1 Notations: summing over repeated indices

Notation. We are familiar with the notation Σ for sums, which saves space:

$$\overrightarrow{OM} = \sum_{i=1}^{3} x_i\vec{e_i}. \tag{1.2}$$

In tensor algebra, it is customary to omit the Σ symbol *when an index is summed over and appears twice and only twice in an expression*:

$$\overrightarrow{OM} = x_i\vec{e_i}. \tag{1.3}$$

The range of the index i is not specified anymore (whereas in Eq. 1.2 it was specified that i ranges from 1 to 3). So, when an index appears twice in an expression, it is implied that it corresponds to a sum over all possible values of this index. Since the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ has three vectors, the sum in Eq. 1.3 is over $i = 1, i = 2, i = 3$.

Remark (mute indices). If an index is summed over, it is called a *mute* index. It takes all possible values, one per term in the sum, and does not *say* anything about the components of the result: the result does not have this index. One can therefore use another symbol for it without changing the value of the sum. For example in $\vec{x} = x_i\vec{e_i}$, the index i is a mute index, and one can as well write $\vec{x} = x_k\vec{e_k}$.

This notation can be used for indices in matrices as well. From linear algebra, you are already familiar with matrices. Matrices have two indices, one for rows and one for columns. Given the basis $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$, and a linear application u from $\mathbf{R}^3$ to $\mathbf{R}^3$, a matrix U can be used to represent the action of u in the basis (e):

$$U = \mathrm{Mat}(u,(e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}$$

$$u\left(\sum_{i=1}^{3} x_i \vec{e_i}\right) = \sum_{j=1}^{3}\left(\sum_{k=1}^{3} U_{jk} x_k\right)\vec{e_j} \tag{1.4}$$

The expression 1.4 contains two indices (j and k) that appear twice each, and are summed over. The sum convention on repeated indices can be applied to rewrite the expression as follows:

$$u(x_i \vec{e_i}) = U_{jk} x_k \vec{e_j}. \tag{1.5}$$

1.2.2 Scalar product, Kronecker symbol

Let $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal basis of $\mathbf{R}^3$. The scalar products between pairs of vectors in the basis is as follows:

$$\vec{e_i}.\vec{e_j} = \delta_{ij}. \tag{1.6}$$

for which one used the special notation δ_{ij} (or Kronecker symbol):

$$\begin{cases} \delta_{ij} = 1 \text{ if } i = j, \\ \delta_{ij} = 0 \text{ if } i \neq j \end{cases}$$

The (Euclidean) scalar product (or dot-product) between two vectors $x = x_i\vec{e_i}$ and $y = y_i\vec{e_i}$ is denoted by a dot, which is a bilinear[4] operation:

$$\vec{x}.\vec{y} = (x_i\vec{e_i}).(y_j\vec{e_j}) = x_i y_j \delta_{ij} = x_i y_i. \tag{1.7}$$

Remark. Whenever repeated indices are summed over, a Kronecker symbol can be inserted (with a new index symbol) without changing the value of the expression:

$$x_i y_i = x_i \delta_{ij} y_j. \tag{1.8}$$

This trick will be used in the next section when we prove that the scalar product is invariant unde change of orthonormal basis.

[4]A bilinear application on a vector space E is an application B with two arguments that is linear in both arguments, meaning for all vectors $\vec{x}$, $\vec{y}$, $\vec{z}$ and all scalars λ and μ we have

$$B(\lambda\vec{x} + \mu\vec{y}, \vec{z}) = \lambda B(\vec{x}, \vec{z}) + \mu B(\vec{y}, \vec{z})$$

and

$$B(\vec{z}, \lambda\vec{x} + \mu\vec{y}) = \lambda B(\vec{z}, \vec{x}) + \mu B(\vec{z}, \vec{y}).$$

In particular, if B is a bilinear map, knowing the values of $B(\vec{e_i}, \vec{e_j})$ for all the vectors in a base is enough to know all the values of B, just as knowing the values of a linear application on the vectors of a basis is enough to characterize the linear application completely.

1.3 Transformation rules from changes of orthonormal basis

1.3.1 Transformation of vectors

Consider two orthonormal bases of $\mathbf{R}^3$, denoted by $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ and $(e') = (\vec{e'_1}, \vec{e'_2}, \vec{e'_3})$. Consider a vector $\vec{x}$ in $\mathbf{R}^3$. Let us write the coordinates of this vector in these two bases:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_i \vec{e'_i}. \tag{1.9}$$

Since the basis (e) is orthonormal, one can decompose the vectors (e'_1, e'_2, e'_3) over this basis by taking dot-products:

$$\forall i \in [1..3], \quad \vec{e'_i} = (\vec{e'_i}.\vec{e_j})\vec{e_j}. \tag{1.10}$$

Let us substitute the expression 1.10 in the expression of $\vec{x}$ in Eq. 1.12:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_i (\vec{e'_i}.\vec{e_j})\vec{e_j}. \tag{1.11}$$

Now we have two expressions for the vector $\vec{x}$ in the same basis (e). Hence the coordinates must be equal. There are two mute indices in the second expression of $\vec{x}$, and we can rewrite the indices in the sum in order to have the symbol $\vec{e_i}$ at the end:

$$\vec{x} = x_i \vec{e_i}, \quad \vec{x} = x'_k (\vec{e'_k}.\vec{e_j})\vec{e_j} = x'_k (\vec{e'_k}.\vec{e_i})\vec{e_i}. \tag{1.12}$$

Let us write the equality of the components of the vector $\vec{x}$:

$$x_i = x'_k (\vec{e'_k}.\vec{e_i}). \tag{1.13}$$

The r.h.s. of Eq. 1.14 can be rewritten as the action of a square matrix denoted by P (the transformation matrix) on the coordinates (x'_1, x'_2, x'_3):

$$\boxed{x_i = P_{ik} x'_k,} \tag{1.14}$$

where

$$\boxed{P_{ik} = \vec{e'_k}.\vec{e_i}.} \tag{1.15}$$

Proposition. *The inverse of the transformation matrix P is its transpose.*

Proof. Consider the product of P and its transpose P^T:

$$(PP^T)_{ij} = P_{ik}(P^T)_{kj} = P_{ik}P_{jk} = (\vec{e'_k}.\vec{e_i})(\vec{e'_k}.\vec{e_j}) = (\vec{e'_l}.\vec{e_i})\delta_{lp}(\vec{e'_p}.\vec{e_j}), \tag{1.16}$$

where we have inserted the Kronecker symbol to rewrite the sum over repeated indices. Let us now rewrite the Kronecker symbol as the dot-product of two vectors of an orthonormal basis:

$$\delta_{lp} = (\vec{e'_l}.\vec{e'_p}). \tag{1.17}$$

Hence

$$(PP^T)_{ij} = (\vec{e'_l}.\vec{e_j})(\vec{e'_l}.\vec{e'_p})(\vec{e'_p}.\vec{e_j}) = (\vec{e_i}.\vec{e'_l})(\vec{e'_l}.\vec{e'_p})(\vec{e_j}.\vec{e'_p}) = \Big((\vec{e_i}.\vec{e'_l})\vec{e'_l}\Big).\Big((\vec{e_j}.\vec{e'_p})\vec{e'_p}\Big) \tag{1.18}$$

where we have used the *bilinearity* of the dot-product. Since (e') is an orthonormal basis, we have

$$\vec{e_i} = (\vec{e_i}.\vec{e'_l})\vec{e'_l}, \text{ and } \vec{e_j} = (\vec{e_j}.\vec{e'_p})\vec{e'_p}. \tag{1.19}$$

Hence

$$\boxed{(PP^T)_{ij} = (\vec{e_i}, \vec{e_j}) = \delta_{ij}.} \tag{1.20}$$

Hence the inverse of the matrix P is its transpose:

$$(PP^T) = I_3, \quad P^TP = I_3. \tag{1.21}$$

We can use this property to obtain an expression of the "new" coordinates x'_1, x'_2, x'_3 in terms of the coordinates x_1, x_2, x_3, simply by multiplying both sides of Eq. 1.14 by P_{ij} (and summing over index i):

$$P_{ij}x_i = P_{ij}P_{ik}x'_k = (P^t)_{ji}P_{ik}x'_k = (P^TP)_{jk}x'_k = \delta_{jk}x'_k = x'_j, \tag{1.22}$$

Hence

$$\boxed{x'_i = P_{ji}x_j.} \tag{1.23}$$

Exercise. Verify that the norm of a vector and the scalar product of two vectors are invariant under a change of orthonormal basis.

Solution. Let us use the same notations as a above. The norm of the vector $\vec{x}$ expressed using the coordinates (x_1, x_2, x_3) is $\sqrt{x_ix_i}$. Using the new coordinates it is expressed as $\sqrt{x'_ix'_i}$. Let us use Eq. 1.14 to write these two expressions in the same system of coordinates:

$$x_ix_i = P_{ik}x'_kP_{il}x'_l \tag{1.24}$$

The expression $P_{ik}P_{il}$ can be rewritten as $(P^T)_{ki}P^T_{il} = (P^TP)_{kl} = \delta_{kl}$. We therefore have

$$x_ix_i = \delta_{kl}x'_kx'_l = x'_kx'_k. \tag{1.25}$$

As the indices i and k are mute in the above expression, the two quantities are equal, and the norm of x is invariant under a change of orthonormal basis.
As for the scalar product of two vectors x and y, we can use the same property to prove the invariance under change of orthonormal basis:

$$x_iy_i = P_{ik}x'_kP_{il}y'_l = (P^T)_{ki}P_{il}x'_ky'_l = (P^TP)_{kl}x'_ky'_l = \delta_{kl}x'_ky'_l = x'_ky'_k. \tag{1.26}$$

1.3.2 Transformation of the matrix of an endomorphism under a change of orthonormal basis

Consider the matrix U of a linear application from $\mathbf{R}^3$ to $\mathbf{R}^3$, presented in an orthonormal basis $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$U = \mathrm{Mat}(u, (e)) = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \tag{1.27}$$

For i in $[1..3]$, the image of vector e_i can be written in the basis (e) using the entries in the i-th row of the matrix U:

$$u(e_i) = U_{i1}\vec{e_1} + U_{i2}\vec{e_2} + U_{i3}\vec{e_3} = U_{ij}\vec{e_j}. \tag{1.28}$$

Consider another orthonormal basis $(e') = (\vec{e'_1}, \vec{e'_2}, \vec{e'_3})$. We want to compute the matrix of u in the basis (e'). Let us call this matrix U':

$$U' = \mathrm{Mat}(u, (e')) = \begin{pmatrix} U'_{11} & U'_{12} & U'_{13} \\ U'_{21} & U'_{22} & U'_{23} \\ U'_{31} & U'_{32} & U'_{33} \end{pmatrix} \tag{1.29}$$

Again, by defnition of the matrix representation of linear applications in basis (e'), we have

$$u(\vec{e'_i}) = U'_{i1}\vec{e'_1} + U'_{i2}\vec{e'_2} + U'_{i3}\vec{e'_3} = U'_{ia}\vec{e'_a}. \tag{1.30}$$

We are going to express both the l.h.s. and the r.h.s of Eq. 1.30 on the basis (e).
The l.h.s of Eq. 1.30 can be rewritten as follows:

$$u(\vec{e'_i}) = u((\vec{e'_i}, \vec{e_a})\vec{e_a}) = (\vec{e'_i}, \vec{e_a})u(\vec{e_a}) = (\vec{e'_i}, \vec{e_a})U_{ak}\vec{e_k} = P_{ai}U_{ak}\vec{e_k}. \tag{1.31}$$

where we used the linearity of u in the second equality, and Eq. 1.28 in the third equality.
The r.h.s of Eq. 1.30 can be rewritten as follows:

$$U'_{ia}\vec{e'_a} = U'_{ia}(\vec{e'_a}, \vec{e_k})\vec{e_k} = U'_{ia}P_{ka}\vec{e_k}. \tag{1.32}$$

As (e) is a basis of $\mathbf{R}^3$, the coefficients of $\vec{e_k}$ in 1.31 and 1.32 must be equal for all k:

$$U'_{ia}P_{ka} = P_{ai}U_{ak}. \tag{1.33}$$

Since the inverse of the transformation matrix P is its transpose P^T, we can multiply both sides by P_{kj} and get rid of all P symbols on the l.h.s.:

$$U'_{ia}P_{ka}P_{kj} = P_{ai}U_{ak}P_{kj}. \tag{1.34}$$

Since $P_{ka}P_{kj} = (P^T)_{ak}P_{kj} = (P^TP)_{aj} = \delta_{aj}$, we have

$$\forall i, j \in [1..3], \quad \boxed{U'_{ij} = P_{ai}P_{kj}U_{ak}.} \tag{1.35}$$

1.4 Definition of tensors

Cartesian tensors are mathematical objects with components in a given orthonormal basis. They carry a certain number of indices (the number of indices is called the order of the tensor), which can be used to act on vectors by summing over indices. When the orthonormal basis is changed, the components of Cartesian tensors are transformed using transformation matrices.

We have seen examples of low order:
- scalars have no indices, they are invariant under change of basis. They are called tensors of order zero (physical examples are mass and temperature). They are called tensors of order 0.
- vectors have one index, they tansform as in Eq. 1.14 under change of basis (physical examples include velocities). They are called tensors of order 1.
- matrices have two indices, they. Physical examples include the stress tensor will be introduced in this course. They are called tensors of order 2.
By generalizing the transformation pattern of Eqs. 1.23 and 1.35, one defines a tensor T of order n to have n indices, with the transformation rule of the entries:

$$\boxed{\forall i_1, i_2, \dots, i_n \in [1..3], \;\; T'_{i_1,\dots,i_n} = P_{k_1 i_1} P_{k_2 i_2} \dots P_{k_n i_n} T_{k_1,\dots,k_n}} \tag{1.36}$$

when one goes from orthonormal basis (e) to orthonormal basis (e'), with the transformation matrix defined by $P_{ij} = e'_i.e_j$.

Exercise. Consider a tensor of order 2 whose components in some orthonormal basis are Kronecker symbols. Show that it has the same coefficients in all orthonormal bases (one says δ is an *isotropic tensor*).

Let us apply the transformation rule of a tensor of order 2, with the above notation for the transformation matrix:

$$(\delta')_{ij} = P_{ki}P_{lj}\delta_{kl} = P_{ki}P_{kj} = (P^T)_{ik}P_{kj} = (P^TP)_{ij} = \delta_{ij}. \tag{1.37}$$

One can also note that the Kronecker symbol is the matrix of the identity transformation, hence its matrix is I_3 in any basis.

Tutorial Sheet 1 : from linear algebra to tensors

Exercise 0. Report by email issues of any kind (typos, inconsistencies, lack of clarity) you found in the lecture notes.

Exercise 1. Transformation of matrices. Consider two orthonormal bases of $\mathbf{R}^3$, and the matrices U and U' of an endomorphism in these two bases. Find the relation between the matrices U and U' in terms of the transformation matrix (section 3.2 of the notes).

Exercise 2. Invariances. Show that the dot-product of two vectors is invariant under a change of orthonormal basis. Show that the trace of a matrix does depend on the basis.

Exercise 3. A geometric example. Consider the case where the basis $(e') = \left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$ is obtained from $(e) = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ by a rotation of axis $\vec{e_3}$ and of angle θ.
a) Draw the two bases, and use trigonometry to write the components of vectors $\left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$ in the basis (e).
b) Consider a vector $\vec{x}$ in the two-dimensional space spanned by $(\vec{e_1}, \vec{e_2})$. Add this vector to your drawing. Call ϕ the angle between $\vec{e_1}$ and $\vec{x}$. Write the components of $\vec{x}$ in the basis (e) and in basis (e'), in terms of the angles θ and ϕ, again using trigonometry (not the formulas from the course).
c) Now use the formulas from the course: write the transformation matrix in terms of θ. Take the coordinates of $\vec{x}$ in the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ (written in terms of ϕ), and apply the transformation rules of tensors of order 1 to find the coordinates of $\vec{x}$ in basis $\left(\vec{e'_1}, \vec{e'_2}, \vec{e'_3}\right)$. Compare with the result of b).

Exercise 4. Euclidean geometry: change of orthonormal basis and orthogonal transformations. Let u be an endomorphism of $\mathbf{R}^N$ (we will be interested in the case $N = 3$, but the results of this exercise hold for any finite dimension N). Let $(e) = (\vec{e_1}, \vec{e_2}, \ldots, \vec{e_N})$ be an orthonormal basis of $\mathbf{R}^N$, and let the scalar product be the bilinear application defined by the relations

$$\forall i, j \in [1..N], \quad \vec{e_i}.\vec{e_j} = \delta_{ij}. \tag{1.38}$$

Define the norm $N(\vec{x})$ of any vector $\vec{x}$ as follows:

$$N : \mathbf{R}^N \longrightarrow \mathbf{R}_+, \quad N(\vec{x}) = \sqrt{\vec{x}.\vec{x}} \tag{1.39}$$

1) Prove that if u preserves the norm, then it preserves the scalar product, meaning: if $N(u(\vec{x})) = N(\vec{x})$ for all vectors $\vec{x}$, then $u(\vec{x}).u(\vec{y}) = \vec{x}.\vec{y}$ for all vectors $\vec{x}$ and $\vec{y}$.
2) Show that u preserves the norm if and only if it maps an orthonormal basis to an orthonormal

base.
3) Show that u preserves the norm if and only the inverse of its matrix U in the orthonormal basis (e) is its transpose, $U^{-1} = U^T$.
4) Consider the special case $N = 3$, two bases (e) and (e') of $\mathbf{R}^3$, and the transformation matrix P described in the course. Interpret this matrix geometrically (to what endomorphism of $\mathbf{R}^3$ does P correspond, and in what base?). Show that it preserves the norm, and deduce that the inverse of P is P^T.

Exercise 5 (exam question, June 2015, one of five questions in the three-hour exam). In this question all tensors are defined on the space $\mathbf{R}^3$ endowed with an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. The position of a point M is described in Cartesian coordinates by a vector $\vec{x} = \overrightarrow{OM} = x_i \vec{e_i}$.

(i) [**10 marks**] Compute the following quantities :
(a) δ_{ii}
(b) $\partial_k x_k$
(c) $\partial_k x_k \partial_j x_j$
(d) $\partial_k x_j \partial_j x_k$

(ii) [**5 marks**] Which of the following expressions (if any) are inconsistent with the sum rule over repeated indices?
(a) $A_i = \epsilon_{ijk} C_j D_k$
(b) $A_i B_k = \epsilon_{ijk} C_j D_k$
(c) $C_{ij} C_{ij} C_{ai}$

(iii) [**5 marks**] Recall the definition of an isotropic Cartesian tensor of rank n (in dimension three). Prove that there are no non-zero isotropic Cartesian tensors of rank one.

Kinematics

Lecture 2 : Kinematics and deformations

Keywords. Kinematics, Lagrangian description of flows, tangent vectors, transformation of line elements, deformation, differential geometry.

Convention. Unless otherwise stated, the sum rule over repeated indices is observed.

In this chapter we will develop the mathematical the description of movements of continuous media (which can be elastic solids or fluids) without studying their causes. This is the kinematics of continuous media. A function of *initial position and time* is used to describe the motion (or *flow*) of the continuous medium. To decide whether the medium is deformed, we have to compute how scalar products of vectors change under the flow (see Fig. 1.1 for a physical example on a solid: we are going to describe mathematically how the grid on the l.h.s. of the figure is deformed).
Technically, the first chapter was entirely based on linear algebra (we learned how the components of a fixed object such as a vector or the matrix of a linear application transform under a change of basis). This chapter will be mostly based on differential geometry: we will work with a fixed basis, but study the tangent vectors to moving curves.

1.5 Lagrangian description of flows

1.5.1 Definitions and notations

To avoid boundary effects, let us study a sample of continuous medium (solid or fluid) that "fills the entire space" (meaning we are describing matter at scales that are large compared to the size of molecules, but we are far away from the boundaries of the sample).

The space $\mathbf{R}^3$ is endowed with an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ (which will be kept fixed in this chapter) and a fixed origin O, so the position of a material point M can be described by its coordinates, $\overrightarrow{OM} = \vec{x} = x_i \vec{e_i}$. We are interested in the way classical particles (small lumps of continuous medium, which could be visualized using tracers in a fluid, or simply by marking the surface by a grid on a solid, as in Fig. 1.1) move over time. Let us say that a particle is at position $\vec{X}$ at time $t = 0$ (one says that $\vec{X}$ is the initial position of the particle). Let us say we are interested in the deformation of the medium between time $t = 0$ and some final time T (the duration of an experiment for instance).

The position of a material particle at time $t > 0$ can be described by a vector which depends on t and on the initial position. Let us denote this vector by $\Phi_1(\vec{X}, t)\vec{e_1} + \Phi_2(\vec{X}, t)\vec{e_2} + \Phi_3(\vec{X}, t)\vec{e_3}$. Since we can consider any initial position $\vec{X}$, we have just defined a mapping $\vec{\Phi}$ that describes the movements of the continuous medium over a time between $t = 0$ and $t = T$:

$$\mathbf{R}^3 \times [0, T] \longrightarrow \mathbf{R}^3 \tag{1.40}$$

$$(\vec{X}, t) \mapsto \vec{\Phi}(\overrightarrow{\vec{X}}, t) = \Phi_1(\vec{X}, t)\vec{e_1} + \Phi_2(\vec{X}, t)\vec{e_2} + \Phi_3(\vec{X}, t)\vec{e_3}. \tag{1.41}$$

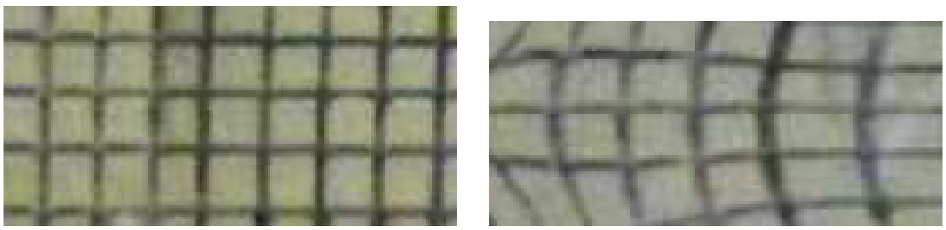

Figure 1.1: **Left.** A sample of perspex with orthogonal lines drawn with a pen at $t = 0$. **Right.** The same sample at some time $t > 0$ after traction.

]

We will assume that the mapping $\vec{\Phi}$ is smooth (it is continuous and has as many continuous derivatives as we need) in all the variables, as we want to study smooth deformations. Continuity w.r.t. time at time $t = 0$ implies that for all $\vec{X}$ in $\mathbf{R}^3$, we have the property $\vec{\Phi}(\vec{X}, 0) = \vec{X}$. In other words the mapping $P\vec{h}i$ is a way to describe the trajectories of all points in the sample (a collection of trajectories).

We want to answer the following question (explicit examples will be given in the exercises): **given the application** $\vec{\Phi}$, and two vectors in the initial state (two arrows drawn at the same point in the initial state), *can we compute the evolution of the scalar product of these two vectors?*

1.5.2 Computing the tangent to a curve in three dimensions

• **Let us parametrize a straigth line at time $t = 0$.** With the system of coordinates described by the origin O and the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, consider the particle which is at position $\vec{X} = O\vec{M}$ in the continuous medium, at time $t = 0$. We can move around this vector by adding some vector $\vec{h} \in \mathbf{R}^3$ to it. We will need to do differential calculus (in order to compute tangent vectors), so let us introduce a scalar parameter $s \in \mathbf{R}$, which we will eventually let go to 0.

If we consider the set of all possible values of the parameter s, we get the straight line going through the point M in direction $\vec{h}$. The vector $\vec{h}$ is the tangent vector of the straight line that goes through M in direction $\vec{h}$, as we have

$$\frac{1}{s}\left((\vec{X} + s\vec{h}) - \vec{X}\right) \longrightarrow_{s \to 0} \vec{h}. \tag{1.42}$$

NB: the parameter s is not time, it is just a geometric parameter describing a straight line at a fixed time ($t = 0$).

- **What happens to this line through time? It is mapped to a line defined in terms of the flow $\vec{\Phi}$.** What does this situation look like at time $t > 0$?

By definition of the flow, the particle that was at $\vec{X}$ is now at $\vec{\Phi}(\vec{X}, t)$ and the particle that was at $\vec{X} + s\vec{h}$ is now at $\vec{\Phi}(\vec{X} + s\vec{h}, t)$, so the quantity we studied in Eq. 1.42 has now become

$$\frac{1}{s}\left(\vec{\Phi}(\vec{X} + s\vec{h}, t) - \vec{\Phi}(\vec{X}, t)\right) \tag{1.43}$$

Let us study its limit when s goes to zero, which is[5] the tangent to the curve $s \in \mathbf{R} \mapsto \vec{\Phi}(\vec{X}+s\vec{h}, t)$.

Let us write all the components in matrix form in the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ to compute this tangent:

$$\vec{\Phi}(\vec{X} + s\vec{h}, t) - \vec{\Phi}(\vec{X}, t) = \begin{pmatrix} \Phi_1(\vec{X} + s\vec{h}, t) - \Phi_1(\vec{X}, t) \\ \Phi_2(\vec{X} + s\vec{h}, t) - \Phi_2(\vec{X}, t) \\ \Phi_3(\vec{X} + s\vec{h}, t) - \Phi_3(\vec{X}, t) \end{pmatrix} \tag{1.44}$$

Let us assume that the components of $\vec{\Phi}$ are smooth functions (they have derivatives at all orders). One can write a first-order Taylor expansion of each of the components of the vector in Eq. 1.44. For example:

$$\Phi_1(\vec{X} + s\vec{h}, t) = \Phi_1(\vec{X}, t) + sh_1\frac{\partial \Phi_1}{\partial X_1}(\vec{X}, t) + sh_2\frac{\partial \Phi_1}{\partial X_2}(\vec{X}, t) + sh_3\frac{\partial \Phi_1}{\partial X_3}(\vec{X}, t) + o(s). \tag{1.45}$$

Hence the quantity

$$\frac{1}{s}\left(\Phi_1(\vec{X} + s\vec{h}, t) - \Phi_1(\vec{X}, t)\right) = h_1\frac{\partial \Phi_1}{\partial X_1}(\vec{X}, t) + h_2\frac{\partial \Phi_1}{\partial X_2}(\vec{X}, t) + h_3\frac{\partial \Phi_1}{\partial X_3}(\vec{X}, t) + \frac{o(s)}{s}, \tag{1.46}$$

and its limit when s goes to zero is given by:

$$h_i\frac{\partial \Phi_1(\vec{X}, t)}{\partial X_i}, \tag{1.47}$$

where the sum rule over repeated indices is applied. The computation can be repeated for Φ_2 and Φ_3, and one obtains the following limit:

$$\frac{1}{s}\left(\vec{\Phi}(\vec{X} + s\vec{h}, t) - \vec{\Phi}(\vec{X}, t)\right) \longrightarrow_{s\to 0} \begin{pmatrix} h_i\frac{\partial \Phi_1(\vec{X},t)}{\partial X_i} \\ h_i\frac{\partial \Phi_2(\vec{X},t)}{\partial X_i} \\ h_i\frac{\partial \Phi_3(\vec{X},t)}{\partial X_i} \end{pmatrix}. \tag{1.48}$$

[5]As an exercise, the reader should put the symbols $\vec{X}, \vec{h}, \vec{\Phi}(\vec{X}, t), \vec{\Phi}(\vec{X} + \vec{h}, t)$ on Fig. 1.1, and draw the two curves we are talking about.

• **A line is mapped to a line, tangent vectors are mapped to tangent vectors.** Hence the vector $\vec{h} = h_j\vec{e_j}$pointing from $\vec{X}$ is mapped to the vector

$$h_i \frac{\partial \Phi_j(\vec{X},t)}{\partial X_i}\vec{e_j} = h_i T_{ji}(\vec{X},t)\vec{e_j}, \tag{1.49}$$

by the flow, where we have defined the matrix T (whose value depends on the initial position $\vec{X}$ by:

$$T_{ij}(\vec{X},t) = \frac{\partial \Phi_i(\vec{X},t)}{\partial X_j}, \tag{1.50}$$

which maps vectors from position $\vec{X}$ to position $\vec{\Phi}(\vec{X},t)$. T is called the *strain tensor*. It depends on the initial point $\vec{X}$ so one could (should) call it *strain tensor field*.

1.6 How does the scalar product change under the flow?

Taking a pair of vectors $\vec{h} = h_i\vec{e_i}$ and $\vec{u} = u_i\vec{e_i}$ starting from position $\vec{X}$, one can transport both vectors by the flow using Eq. 1.49, and compute the dot-product of the resulting vectors:

$$\begin{aligned}\left(h_i \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i}\vec{e_j}\right) . \left(u_k \tfrac{\partial \Phi_l(\vec{X},t)}{\partial X_k}\vec{e_l}\right) &= h_i u_k \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i}\tfrac{\partial \Phi_l(\vec{X},t)}{\partial X_k}\delta_{jl} \\ &= h_i u_k \tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_i}\tfrac{\partial \Phi_j(\vec{X},t)}{\partial X_k} \\ &= h_i u_k T_{ji}(\vec{X},t) T_{jk}(\vec{X},t).\end{aligned} \tag{1.51}$$

One could be tempted to say that there is "no deformation" if the matrix T is the identity matrix. However this condition is too restrictive. Indeed we can see from Eq. 3.2 that if $T_{ji}T_{jk} = \delta_{ik}$, the scalar product is preserved (hence the grid drawn on Fig. 1.1 is not deformed, just transported, as is the case for a rotation of the sample, for instance, see Tutorial Sheet 2, Exercise 1). Hence we see that the local deformations are measured by the measured by the difference $T_{ji}T_{jk} - \delta_{ik}$ (see Tutorial Sheet 2, Exercise 2 for such a situation).

Tutorial Sheet 2 : Kinematics and deformations

In the following exercises, one describes a sample continuous medium, far from the boundaries (which is equivalent to considering that the sample fills the entire space). The space $\mathbf{R}^3$ is endowed with an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, and a fixed origin O, so that the position of a point M can be described by the coordinates (x_1, x_2, x_3), such that $\vec{OM} = x_i \vec{e_i}$. **The sum rule over repeated indices in used.**

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. A flow without deformation. Write the flow $\vec{\Phi}$ that corresponds to a rotation of a continuous medium around an axis parallel to $\vec{e_3}$ passing through O (by ω radians per unit of time). Using this flow and the transformation rules of the dot-products at the end of the notes, show that there is no deformation (i.e. that all dot-products of pairs of vectors are conserved by the flow).

Exercise 2. The marble-cake deformation. Consider the following flow (describing the motion of a continuous medium filling the entire space $\mathbf{R}^3$:

$$\vec{\Phi}(\vec{x}, t) = \begin{pmatrix} x_1 \\ x_2 + Vt \sin\left(\frac{2\pi x_1}{L}\right) \\ x_3 \end{pmatrix}, \tag{1.52}$$

Interpret the parameters V and L physically (what is their unit?).
1) Check that at time $t = 0$, the flow is the identical application, i.e. that for all $\vec{x} \in \mathbf{R}^3$, one has $\vec{\Phi}(\vec{x}, 0) = \vec{x}$.
2) Consider the cube $\{[x_1, x_2, x_3], 0 \leq x_1 \leq L, 0 \leq x_2 \leq L, 0 \leq x_3 \leq L\}$. Draw a cross section of the cube through the plane $x_3 = 0$, with a regular grid drawn on it (as on the left of Fig. 1 in the notes), at time $t = 0$. Draw this cross section with the grid at some time $t > 0$.
3) Consider the point P of coordinates $(x_1 = L/2, x_2 = L/2, x_3 = 0)$ at time 0. Draw it on the initial cross section.
Compute its image $\vec{\vec{\Phi}}(L/2, L/2, 0, t)$ and draw it on the final cross section.
4) Compute the strain tensor at every point at time t. Give examples of deformations by computing various transformations of scalar products.
5) Consider two unit vectors $\vec{e_1}$ and $\vec{e_2}$ pointing from P at $t = 0$. How do the dot-products $\vec{e_1}.\vec{e_1}$, $\vec{e_2}.\vec{e_2}$ and $\vec{e_1}.\vec{e_2}$ transform under the flow? At fixed t, are there points with no deformation?

Chapter 2

Dynamics: the stress tensor

Lecture 3: Static equilibrium and the stress tensor in the bulk

2.1 Body forces and surface forces, the Cauchy postulate

A *volume force* (or *body force*) is a force that acts acts on a sample of matter, and is proportional to the volume of the sample. Gravity is an example of volume force. On an infinitesimally small volume dV centered[1] at point $\vec{x}$, a volume force is written as the product of dV by the intensity $\overrightarrow{f^{vol}}(\vec{x})$ of the force per unit volume:

$$\overrightarrow{f^{vol}}(\vec{x})dV. \tag{2.1}$$

I want to refer to my last Eq. 2.1.

In the case of gravity, $\overrightarrow{f^{vol}}(\vec{x}) = \rho(\vec{x})\vec{g}$, where ρ is the density of the material, and $\vec{g}$ is the gravitational field.

A *surface force* is a force acting on the surface of a sample of matter, which is proportional to the area of the surface. Cauchy postulated (in the early 19th century) that the forces that maintain the cohesion of a continuous medium are surface forces. On a small surface[2] centered at point $\vec{x}$, with a unit normal vector $\vec{n}$, such a force is written as

$$\vec{T}(\vec{x},\vec{n})dS, \tag{2.2}$$

expressing the fact the force is proportional to the surface, depends on its location (through the variable $\vec{x}$) and on its orientation (through the variable $\vec{n}$). Let us consider a continuous medium at equilibrium occupying the whole space (we are not looking at boundaries yet), and isolate (in thought) a domain $\mathcal{V}$ with boundary $\mathcal{S}$ (oriented towards the exterior of the volume, meaning the normal vector at every point of the surface points towards the exterior of the domain). If the domain $\mathcal{V}$ is at equilibrium, the volume and surface forces add up to zero:

$$\boxed{\vec{0} = \iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{x})dV + \oiint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{x},\vec{n})dS.} \tag{2.3}$$

[1] in this chapter, we take a coordinate system on $\mathbf{R}^3$ with an origin O and an orthonormal base $(\vec{e_1},\vec{e_2},\vec{e_3})$, so we can identify the vector $\vec{x} = x_i\vec{e_i}$ to the point M such that $\overrightarrow{OM} = x_i\vec{e_i}$. Moreover, we use the sum rule over repeated indices.

[2] By *unit normal vector* one means a vector that is orthogonal to the surface and has norm 1, i.e. if $(\vec{e_1},\vec{e_2},\vec{e_3})$ is an orthonormal base and $\vec{n} = n_i\vec{e_i}$, one has $n_in_i = 1$.

2.2 Stokes' theorem: reminder of the scalar version, tensor version

From calculus and physics you are probably familiar with Stokes' theorem written in the following form:

$$\iiint_{\mathcal{V}} \frac{\partial \phi_i(\vec{x})}{\partial x_i} dV = \oiint_{\mathcal{S}\to\text{ext}} \phi_i(\vec{x}) n_i(\vec{x}) dS \tag{2.4}$$

where $\vec{\phi}$ is a vector field (meaning there is a vector $\vec{\phi}(\vec{x})$ at every point $\vec{x}$ in space).

This can be generalised to higher-order tensors. For example, if τ is a tensor with two indices, the following equations (one equation per value of j) hold:

$$\forall j \in \{1,2,3\}, \quad \iiint_{\mathcal{V}} \frac{\partial \tau_{ji}(\vec{x})}{\partial x_i} dV = \oiint_{\mathcal{S}\to\text{ext}} \tau_{ji}(\vec{x}) n_i(\vec{x}) dS. \tag{2.5}$$

Fixing index j, one can just apply Stokes' theorem to the vector $\vec{\phi}$ with components $\phi_1 = \tau_{j1}$, $\phi_2 = \tau_{j2}$, $\phi_3 = \tau_{j3}$, and obtain Eq. 2.5. See the discussion of the Cauchy tetrahedron in the next section for a geometric application.

2.3 Surface forces are a linear function of the normal vector to the surface

In this section we are going to expose the argument due to Cauchy (about 1820), to prove that the vector $\vec{T}(\vec{x}, \vec{n})$ is a linear function of the normal vector $\vec{n} = n_i \vec{e_i}$, meaning:

$$\vec{T}(\vec{x}, \vec{n}) = n_i \vec{T}(\vec{x}, \vec{e_i}). \tag{2.6}$$

For definiteness let us consider that the volume forces consist of gravity:

$$\overrightarrow{f^{vol}}(\vec{x}) = \rho(\vec{x})\vec{g}, \tag{2.7}$$

where ρ is the density of the continuous medium (note that it depends on the position $\vec{x}$, since the continuous medium is not necessarily homogeneous), but the result would hold for any bounded volume force.

2.3.1 Balance equation of a thin cylinder

First we can establish the following lemma:

$$\vec{T}(\vec{x}, -\vec{n}) = -\vec{T}(\vec{x}, \vec{n}). \tag{2.8}$$

Let us consider a cylinder of height l, and of radius a, cut by a thought experiment inside a continuous medium as in Fig. 2.1. Let us say that the center of the upper disk is at point M,

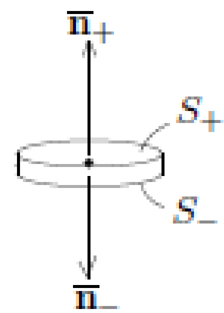

Figure 2.1: A thin cylinder at equilibrium.

with $\overrightarrow{OM} = \vec{x}$, and that the unit normal vector to the upper disk pointing towards the exterior coincides with $\vec{n}$. Let $\vec{n}_+$ and $\vec{n}_-$ denote the unit normal vectors to the upper and lower disks. These two unit vectors are opposite to each other:

$$\vec{n}_+ = -\vec{n}_-. \tag{2.9}$$

If a is small enough (but fixed), the dependence of $\vec{T}$ in its first argument can be neglected, the density can be considered uniform (as $\vec{T}$ and ρ are assumed to be smooth functions), and the equilibrium of the cylinder inside the continuous medium are written as:

$$\vec{0} = \rho\vec{g}\pi a^2 l + \left(\vec{T}(\vec{x},\vec{n}_+) + \vec{T}(\vec{x} - l\vec{n}_+, \vec{n}_-)\right)\pi a^2 + \oiint_{lateral} \vec{T}(\vec{x}, \vec{n}_{lateral})dS, \tag{2.10}$$

where the last term in the r.h.s is the integral over the lateral area of the cylinder, which goes to $\vec{0}$ when l goes to zero (because the integrand is bounded and the lateral surface goes to zero). The first term also goes to zero when l goes to zero, whereas the second term (which is the integral of forces on surfaces S_+ and S_- of Fig. 2.1) goes to $\left(\vec{T}(\vec{x},\vec{n}_+) + \vec{T}(\vec{x},\vec{n}_-)\right)\pi a^2$. As $\vec{n}_+ = -\vec{n}_-$, the limit of the equilibrium condition of the cylinder when l goes to zero becomes:

$$\vec{T}(\vec{x}, -\vec{n}_+) = -\vec{T}(\vec{x}, \vec{n}_+), \tag{2.11}$$

which is just Eq. 2.8.

2.3.2 Balance equation of a (small) tetrahedron

Let us call M the point described by vector $\vec{x}$ in our reference system: $\overrightarrow{OM} = \vec{x} = x_i\vec{e_i}$. Let us consider a tetrahedron with one corner at M, and the three other corners P_1, P_2, P_3, such that the vectors $\overrightarrow{MP_i}$ are parallel to the axes of the orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$\overrightarrow{MP_1} = \epsilon y_1\vec{e_1}, \quad \overrightarrow{MP_2} = \epsilon y_2\vec{e_2}, \quad \overrightarrow{MP_3} = \epsilon y_3\vec{e_3}, \tag{2.12}$$

and ϵ To ease calculations, the three parameters y_i, for i in $\{1, 2, 3\}$ have been chosen to be positive (this can always be ensured upon choosing the axes, see Fig. 2.2). The parameter ϵ is a positive number which we will eventually let go to zero[3], to use the fact that the volume

[3] ϵ has no unit, it is just a number, so when ϵ goes to zero the tetrahedron becomes very small, and all its points become very close to M, but the tetrahedron keeps the same shape.

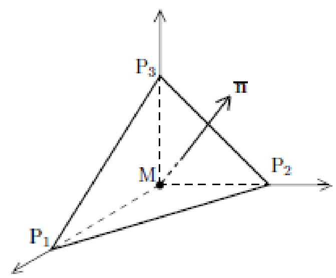

Figure 2.2: The Cauchy tetrahedron.

forces scale as ϵ^3, while the surface forces scale as ϵ^2. The tetrahedron is filled by the continuous medium and it is at equilibrium, so if we take $\mathcal{V}$ to be the tetrahedron and $\mathcal{S}$ to be the surface of the tetrahedron, we have the following equation:

$$\iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{y})dV + \oiint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{y},\vec{n}(\vec{y}))dS = \vec{0}. \tag{2.13}$$

As $\vec{x}$ is reserved as a notation for the fixed vector $\overrightarrow{OM} = \vec{x}$ in this section, we use $\vec{y}$ as the integration variable (describing the position of points inside the tetrahedron). To complete the argument we need to do some geometry.

The surface integral consists of four terms, one per face of the tetrahedron. The unit normal vectors (oriented towards the exterior of the tetrahedron) to the faces MP_1P_2, MP_2P_3 and MP_1P_3 are the vectors $-\vec{e_3}$, $-\vec{e_1}$ and $-\vec{e_2}$ respectively, and the areas of these faces are $\epsilon^2(y_1y_2)/2$, $\epsilon^2(y_2y_3)/2$ and $\epsilon^2(y_1y_3)/2$ respectively. Let us denote by $\vec{n} = n_i\vec{e_i}$ the unit normal vector to the oblique face $P_1P_2P_3$ oriented towards the exterior of the tetrahedron (see Fig. 2.2), and by S_ϵ its area. By applying Pythagoras theorem one can show that $S_\epsilon = \epsilon^2 S(1)$ (where $S(1)$ is the value of the area of the oblique face when $\epsilon = 1$, and depends only on y_1, y_2, y_3).

Let us denote by C the center of the oblique face $P_1P_2P_3$, and by C_3, C_1, C_2 the centers of faces MP_1P_2, MP_2P_3 and MP_1P_3. The volume $V_{MP_1P_2P_3}$ of the tetrahedron equals a third of the product of the surface of the base by the height (the numerical factor of 1/3 is not so important, we will be mostly interested in the power of ϵ):

$$V_{MP_1P_2P_3} = \frac{1}{3} \times \frac{\epsilon y_1 \times \epsilon y_2}{2} \times \epsilon y_3 = \epsilon^3 \frac{y_1y_2y_3}{6}, \tag{2.14}$$

We are going to use the continuity of $\overrightarrow{f^{vol}}$ and $\vec{T}$ w.r.t. the variable $\vec{x}$ (all forces are always assumed to have derivatives, in particular they are continuous), as the points C, C_1, C_2 and C_3 all go to M when ϵ goes to zero. At leading order in ϵ, the volume term in the balance equation is therefore:

$$\iiint_{\mathcal{V}} \overrightarrow{f^{vol}}(\vec{y})dV = \overrightarrow{f^{vol}}(\overrightarrow{OM})\frac{y_1y_2y_3}{6}\epsilon^3 + o(\epsilon^3), \tag{2.15}$$

while the surface term reads

$$\oiint_{\mathcal{S}\to\text{ext}} \vec{T}(\vec{y},\vec{n})dS = \left(\vec{T}(\overrightarrow{OM},\vec{n})S(1) + \vec{T}(\overrightarrow{OM},-\vec{e_1})\frac{y_2y_3}{2} + \vec{T}(\overrightarrow{OM},-\vec{e_2})\frac{y_1y_3}{2} + \vec{T}(\overrightarrow{OM},-\vec{e_3})\frac{y_1y_2}{2}\right)\epsilon^2 + o \tag{2.16}$$

Because the normal vector $\vec{n}$ does not depend on ϵ. So the leading order in ϵ of the balance equation is just a linear combination of values of the function $\vec{T}$ at the same point in space, but different value of the normal vector:

$$\vec{T}(\overrightarrow{OM},\vec{n})S(1) + \vec{T}(\overrightarrow{OM},-\vec{e_1})\frac{y_2y_3}{2} + \vec{T}(\overrightarrow{OM},-\vec{e_2})\frac{y_1y_3}{2} + \vec{T}(\overrightarrow{OM},-\vec{e_3})\frac{y_1y_2}{2} = \vec{0}. \tag{2.17}$$

Let us express the vector $\vec{n}$ in terms of our geometric data. We can apply Stokes' theorem (Eq. 2.5) to the Kronecker symbol: $\tau_{ij}(\vec{x}) = \delta_{ij}$ (the value is uniform, independent from $\vec{x}$), and to the tetrahedron. The integral over the volume is zero because the tensor has no dependence over the point, and the surface integral is just the integral of the normal vector over the surface, which can readily be expressed in terms of our notations:

$$\vec{0} = S(1)\vec{n} - \frac{1}{2}\left(y_2y_3\vec{e_1} + y_1y_3\vec{e_2} + y_1y_2\vec{e_3}\right). \tag{2.18}$$

Taking dot-products of this equation with vectors $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$, one obtains the following relations between the components of the vector $\vec{n}$ and the geometric parameters of the problem:

$$S(1)n_1 = \frac{1}{2}y_2y_3, \quad S(1)n_2 = \frac{1}{2}y_1y_3, \quad S(1)n_3 = \frac{1}{2}y_1y_2. \tag{2.19}$$

Hence we can rewrite Eq. 2.17, letting components of $\vec{n}$ appear:

$$\vec{0} = \vec{T}(\overrightarrow{OM},\vec{n})S(1) + \left(\vec{T}(\overrightarrow{OM},-\vec{e_1})S(1)n_1 + \vec{T}(\overrightarrow{OM},-\vec{e_2})S(1)n_2 + \vec{T}(\overrightarrow{OM},-\vec{e_3})S(1)n_3\right) \tag{2.20}$$

Dividing by $S(1)$ and applying Eq. 2.8 we obtain:

$$\vec{0} = \vec{T}(\overrightarrow{OM},\vec{n}) - \vec{T}(\overrightarrow{OM},\vec{e_1})n_1 - \vec{T}(\overrightarrow{OM},\vec{e_2})n_2 - \vec{T}(\overrightarrow{OM},\vec{e_3})n_3. \tag{2.21}$$

Remembering that $\vec{x} = \overrightarrow{OM}$, we can rewrite this as:

$$\vec{T}(\vec{x},\vec{n}) = n_j\vec{T}(\vec{x},\vec{e_j}), \tag{2.22}$$

which is the linearity in the vector $\vec{n}$ we wanted to establish. The matrix of the linear application is called the stress tensor (it depends on the point $\vec{x}$, so it can be called a tensor *field*). It is denoted by $\sigma(\vec{x})$:

$$\boxed{\vec{T}(\vec{x},n_j\vec{e_j}) = \sigma_{ij}(\vec{x})n_j\vec{e_i}.} \tag{2.23}$$

2.4 Balance equations for a continuous medium

2.4.1 Forces sum to zero

Even though we have started from an equilibrium equation consisting of a volume integral and a surface integral, we can rewrite it as a volume integral thanks to the linearity of force surfaces in the normal (the existence of the stress tensor), and Stoke's theorem. Indeed, Eq. 2.23 allows us to rewrite the equilibrium condition of Eq. 2.3 as

$$\iiint_{\mathcal{V}} f_i^{vol}(\vec{x})dV + \iint_{\mathcal{S}\to\text{ext}} \sigma_{ij}(\vec{x})n_j(\vec{x})dS = 0, \tag{2.24}$$

where $f_i^{vol} = \overrightarrow{f^{vol}}.\vec{e_i}$ is the component of the volume force density along vector $\vec{e_i}$, and Stoke's theorem yields

$$\iiint_{\mathcal{V}} \left(f_i^{vol}(\vec{x}) + \frac{\partial \sigma_{ij}}{\partial x_j}(\vec{x}) \right) dV \vec{e_i} = \vec{0}. \tag{2.25}$$

As the above equation holds for any volume V, the integrand is zero, and the following equilibrium equations hold:

$$\boxed{f_i^{vol} + \frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \forall i \in \{1, 2, 3\}} \tag{2.26}$$

2.4.2 Momenta of forces sum to zero, hence the stress tensor is symmetric

Let us express write that the moments of volume forces and surface forces at the origin O sum to zero:

$$\iiint_{\mathcal{V}} \overrightarrow{OM} \wedge \overrightarrow{f^{vol}}(\vec{x})dV + \oiint_{\mathcal{S}\to\text{ext}} \overrightarrow{OM} \wedge \vec{T}(\overrightarrow{OM}, \vec{n})dS = \vec{0}. \tag{2.27}$$

Let us denote by ϵ the order-three tensor that is totally antisymmetric in all its indices[4] and has $\epsilon_{123} = 1$ (**NB**: in the context of this section, the quantity ϵ, which is an order-three tensor, is not to be confused with the scaling parameter used in the context of the Cauchy tetrahedron, which is a single number, and was denoted by the same symbol). One can easily check as an exercise that the components of the vector product in three dimensions are expressed by taking the tensor product of vectors with the totally antisymmetric tensor:

$$(x_j\vec{e_j}) \wedge (y_k\vec{e_k}) = (\epsilon_{ijk}x_jy_k)\vec{e_i}. \tag{2.28}$$

This enables us to rewrite the integrands of Eq. 2.27 in terms of the components x_i of the vector $\overrightarrow{OM} = x_i\vec{e_i}$. The i-th component of Eq. 2.27 takes the following form:

$$\iiint_{\mathcal{V}} \epsilon_{ijk}x_jf_k^{vol}(\vec{x})dV + \oiint_{\mathcal{S}\to\text{ext}} \epsilon_{ijk}x_j\sigma_{kl}(\vec{x})n_ldS = 0. \tag{2.29}$$

[4]meaning for all values of indices i, j, k, any permutation of two indices flips the sign of the entry: $\epsilon_{ijk} = -\epsilon_{jik}$, $\epsilon_{ijk} = -\epsilon_{ikj}$, $\epsilon_{ijk} = -\epsilon_{jik}$, which implies that all entries of ϵ equal 0 (when at least two indices are equal), 1 (if the indices are obtained from $(1, 2, 3)$ by an even number of permutations), or -1 (if the indices are obtained from $(1, 2, 3)$ by an odd number of permutations).

Stokes' theorem can be applied to the surface integral:

$$\iiint_{\mathcal{V}} \left(\epsilon_{ijk} x_j f_k^{vol}(\vec{x}) + \frac{\partial}{\partial x_l} \left(\epsilon_{ijk} x_j \sigma_{kl}(\vec{x}) \right) \right) dV = 0. \tag{2.30}$$

The derivative w.r.t. x_l gives rise to two terms:

$$\iiint_{\mathcal{V}} \left(\epsilon_{ijk} x_j f_k^{vol}(\vec{x}) + \epsilon_{ijk} \delta_{jl} \sigma_{kl}(\vec{x}) + \epsilon_{ijk} x_j \frac{\partial \sigma_{kl}(\vec{x})}{\partial x_l} \right) dV = 0, \tag{2.31}$$

and the equilibrium of forces (Eq. 2.26) can be used to reduce the integrand to just one term:

$$0 = \iiint_{\mathcal{V}} \left(\epsilon_{ijk} x_j \left(f_k^{vol}(\vec{x}) + \frac{\partial \sigma_{kl}(\vec{x})}{\partial x_l} \right) + \epsilon_{ijk} \sigma_{kj}(\vec{x}) \right) dV = \iiint_{\mathcal{V}} \epsilon_{ijk} \delta_{jl} \sigma_{kl} dV. \tag{2.32}$$

Since this is true for any domain $\mathcal{V}$, the integrand must be zero at every point $\vec{x}$, hence:

$$\epsilon_{ijk} \sigma_{kj}(\vec{x}) = 0 \tag{2.33}$$

This equation implies that σ is symmetric. By the definition of the antisymmetric three-tensor ϵ, if one considers $i = 1$ for instance, Eq. 2.33 only has terms corresponding to $j, k \in \{2, 3\}$, and becomes $\sigma_{23} - \sigma_{32} = 0$. For the same reason, considering $i = 2$ gives rise to $-\sigma_{13} + \sigma_{13} = 0$, and considering $i = 3$ gives rise to $\sigma_{23} - \sigma_{32} = 0$. So we have established the following property as a a consequence of the equilibrium of moments

$$\boxed{\sigma_{ji} = \sigma_{ij}, \quad \forall i, j \in [1..3].} \tag{2.34}$$

Tutorial Sheet 3: Stokes' theorem

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. (i) Prove Stokes' theorem in the case where the integration domain is a cube, with sides parellel to the three directions of a chosen orthonormal base. Observe that it is important to have the boundary oriented towards the exterior.

Hint. Stokes' theorem is a generalization to higher number of variables and higher dimension of the relation

$$f(b) - f(a) = \int_a^b f'(x)dx, \tag{2.35}$$

which holds for smooth functions $f : \mathbf{R} \longrightarrow \mathbf{R}$ (one variable, one-dimensional target space).

Let $\vec{\Phi} : \vec{x} \mapsto \vec{\Phi}(\vec{x}) = \Phi_i(x_1, x_2, x_3)\vec{e_i}$ be a smooth function from $\mathbf{R}^3$ to $\mathbf{R}^3$ (three variables, three-dimensional target space). For computations we will use the three functions Φ_1, Φ_2 and Φ_3, which are functions of three variables with values in $\mathbf{R}$).

Consider a cube $\mathcal{C}$ of side L. Let us put the origin of the coordinate system at one corner of the cube, and choose an orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ with vectors aligned with the axes of the cube, so that the cube is the following domain:

$$\mathcal{C} = \{x_i\vec{e_i}, \quad 0 \leq x_1 \leq L, \quad 0 \leq x_2 \leq L, \quad 0 \leq x_3 \leq L\}\,. \tag{2.36}$$

Let $\partial\mathcal{C}$ denote the boundary of the cube, which consists of six square faces. Start by computing the following quantity, which is the r.h.s. of the equation in Stokes' theorem:

$$\rho = \oiint_{\partial\mathcal{C}\to\text{ext}} \vec{\Phi}(\vec{x}).\vec{n}(\vec{x})dS. \tag{2.37}$$

(ii) What result do you obtain if you put two such cubes side by side?

Exercise 2. Generalise to higher-order tensors (the order-two case is Eq. 5 in the notes).

Lecture 4: Statically admissible stress tensors

So far we only considered continuous media that fill *the entire space*. We studied forces on *test surfaces* inside the continuous medium. We concluded that there exists a stress tensor $\sigma(\vec{x})$ at every point inside a continuous medium, and we related σ to the body forces via a system of three PDEs.

We want to extend this analysis to media with a boundary, and with external forces applied to this boundary (external forces). This will give us boundary conditions for the PDEs. This lecture will enable us to decide whether a given stress tensor $\sigma(\vec{x})$ is compatible with balance equations for a given configuration of volume **and external surface forces**.

2.5 An example of stress tensor: the hydrostatic pressure

Consider a static ocean, submitted to gravity on the surface of the Earth (neglect the curvature of the Earth), assume that the density of water is uniform ($\rho(\vec{x}) \simeq 10^3$ kg.m^{-3}). Hence the body forces per unit volume in the ocean are expressed as:

$$\overrightarrow{f^{vol}} = -\rho g \vec{e_3}. \tag{2.38}$$

where $g \simeq 9.8\,\mathrm{m.s}^{-2}$, and $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is an orthonormal base with $\vec{e_3}$ the vertical vector pointing upwards.

The force acting on a closed test surface inside the ocean (think of a cube) is normal to the surface (hence the name pressure), and does not depend on the orientation of the surface, which we express by assuming there exists a positive function P such that the stress tensor σ has the following form (such an isotropic diagonal tensor is sometimes called a hydrostatic tensor):

$$\sigma_{ij}(\vec{x}) n_j(\vec{x}) \vec{e_i} = -P(\vec{x}) \vec{n}, \tag{2.39}$$

where P is the pressure of the water, a positive function which we have to compute.

In tensor notations we have $\sigma_{ij}(\vec{x} = -P(\vec{x}\delta_{ij}$ (for all $\vec{x}$ in the ocean, and for all indices i and j in $\{1, 2, 3\}$), hence the equilibrium equation condition at every point in the ocean:

$$\overrightarrow{f^{vol}} + \frac{\partial \sigma_{ij}}{\partial x_j} \vec{e_i} = \vec{0}. \tag{2.40}$$

The three components of this vector equation are the following:

$$\begin{cases} 0 &= -\frac{\partial P}{\partial x_1} \\ 0 &= -\frac{\partial P}{\partial x_2} \\ 0 &= -\rho g - \frac{\partial P}{\partial x_3} \end{cases}$$

The first two equations imply that P only depends on x_3, and the third equation can be solved immediatly to show that the pressure is a linear function of the depth, but we need an integration constant K, which is the value of the pressure at the surface of the water:

$$P(x_1, x_2, x_3) = -\rho g x_3 + K. \tag{2.41}$$

To characterize the stress tensor completely, even in this very simple case, we need boundary conditions expressing the equilibrium of our continuous medium at the interface with other media (the atmosphere in this case), that can act on the medium through surface forces. In the next section we address this problem in full generality.

2.6 Boundary conditions

In a more general setting, let $\mathcal{V}$ be a volume occupied by a contiunous medium. Its boundary is denoted by $\partial\mathcal{V}$. On part of the boundary, surface forces can be applied by an operator. Let $\vec{F}(\vec{x})$ denote the force by surface unit applied at a point $\vec{x}$ on the boundary $\partial\mathcal{V}$. Let us denote by $\partial_F\mathcal{V}$ the domain of $\partial\mathcal{V}$ where $\vec{F}$ is defined, **always assumed to be oriented towards the exterior of the continuous medium**. The balance between volume forces and surface forces on $\mathcal{V}$ is expressed in global form as follows:

$$\vec{0} = \iiint_{vol} \overrightarrow{f^{vol}}(\vec{x})dV + \iint_{\partial\mathcal{V}-\partial_F\mathcal{V}} \sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e}_i dS + \iint_{\partial_F\mathcal{V}} \vec{F}(\vec{x})dS, \tag{2.42}$$

where $\vec{n}(\vec{x})$ is the unit normal vector to the surface $\partial\mathcal{V}$ at $\vec{x}$ pointing towards the exterior. Let us add and subtract the integral of $\sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e}_i$ on $\partial_F\mathcal{V}$, which allows us to recognize an integral term on the closed surface $\partial\mathcal{V}$:

$$\begin{aligned} \vec{0} = & \iiint_{\mathcal{V}} \overrightarrow{f^{vol}} dV + \iint_{\partial\mathcal{V}-\partial_F\mathcal{V}} \sigma_{ij}n_j\vec{e}_i dS + \iint_{\partial_F\mathcal{V}} \sigma_{ij}n_j(\vec{x})\vec{e}_i dS - \iint_{\partial_F\mathcal{V}} \sigma_{ij}n_j\vec{e}_i dS + \iint_{\partial_F\mathcal{V}} \vec{F} dS \\ = & \iiint_{\mathcal{V}} \overrightarrow{f^{vol}} dV + \oiint_{\partial\mathcal{V}\to\mathrm{ext}} \sigma_{ij}n_j\vec{e}_i dS - \iint_{\partial_F\mathcal{V}} \sigma_{ij}n_j\vec{e}_i dS + \iint_{\partial_F\mathcal{V}} \vec{F} dS, \end{aligned}$$

where for brevity the dependences of the integrands in $\vec{x}$ have not been written. The first two terms sum to zero, as they express the equilibrium of the continuous medium in $\mathcal{V}$, with an infinitely thin slice of continuous medium covering it on $\partial_F\mathcal{V}$:

$$\vec{0} = \iiint_{\mathcal{V}} \overrightarrow{f^{vol}} dV + \oiint_{\partial\mathcal{V}\to\mathrm{ext}} \sigma_{ij}n_j\vec{e}_i dS, \tag{2.43}$$

hence the equilibrium of $\mathcal{V}$ with the surface force $\vec{F}$ is written as

$$\vec{0} = -\iint_{\partial_F\mathcal{V}} \sigma_{ij}n_j\vec{e}_i dS + \iint_{\partial_F\mathcal{V}} \vec{F} dS. \tag{2.44}$$

As this equation holds for any subset $\partial_F\mathcal{V}$ of $\partial\mathcal{V}$, it holds in the limit where $\partial_F\mathcal{V}$ is an elementary surface centered on $\vec{x}$, and we obtain the local boundary condition:

$$\sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e_i} = \vec{F}(\vec{x}), \quad \forall \vec{x} \in \partial_F\mathcal{V}. \tag{2.45}$$

Example (hydrostatic pressure, continued). Let us come back to our problem of hydrostatic pressure in the sea. The boundary is the entire horizontal surface of equation $x_3 = 0$ at equilibrium with the atmosphere:

$$\partial\mathcal{V} = \left\{(x_1, x_2, 0), x_1 \in \mathbf{R}, x_2 \in \mathbf{R}\right\}. \tag{2.46}$$

Let us assume the atmospheric pressure P_{atm} is uniform on the surface ($P_{atm} \simeq 10^5$ Pa) for instance. The force on the sea per unit surface of the boundary is therefore:

$$\vec{F} = -P_{atm}\vec{e_3}. \tag{2.47}$$

On the boundary, the unit normal vector pointing towards the exterior of the sea is $\vec{n} = \vec{e_3}$. The boundary conditions 2.45 are therefore expressed as follows:

$$-(-\rho g \times 0 + K)\vec{e_3} = -P_{atm}\vec{e_3}. \tag{2.48}$$

i.e. $K = P_{atm}$ and we find that the expression:

$$P(x_1, x_2, x_3) = -\rho g x_3 + P_{atm}, \tag{2.49}$$

and we notice that the pressure is continuous at the surface (which we could have written on physical grounds without developing, this general formalism but we can see in this very simple case that our tensor-based machinery gives the correct results).

2.7 General conclusion: statically admissible stress tensors

We can now summarize our balance laws on continuous media in terms of conditions on the stress tensor. If a continuous medium occupies a volume $\mathcal{V}$ and if the following forces act on it:
- body forces $\overrightarrow{f^{vol}}$ per unit of volume in $\mathcal{V}$,
- specified surface forces $\vec{F}$ per unit of surface on a subset $\partial_F\mathcal{V}$ (the forces may not be specified on the entire boundary $\partial\mathcal{V}$), then the continous medium is at static equilibrium if the stress tensor verified the following equations:

$$\begin{cases} \vec{0} & = \overrightarrow{f^{vol}}(\vec{x}) + \frac{\partial\sigma_{ij}(\vec{x})}{\partial x_j}\vec{e_i}, \quad \forall \vec{x} \in \mathcal{V} \\ \vec{F}(\vec{x}) & = \sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e_i}, \quad \forall \vec{x} \in \partial_F\mathcal{V} \end{cases}$$

where $\vec{n} = n_j(\vec{x})\vec{e_j}$ is the normal vector to the boundary of $\mathcal{V}$, pointing towards the exterior.
These equations (PDEs with boundary conditions) express the fact that the sum of forces on

the medium equal zero. Moreover, the tensor σ is symmetric (which ensures that the moments of forces sum to zero).

The set of *statically admissible stress tensors* associated to this geometry and these forces is defined as the set of (fields of) three-by-three symmetric matrices satisfying the PDE expressing that the forces surm to zero, together with the boundary conditions:

$$\boxed{\begin{aligned}\Sigma_{stat}(\mathcal{V}, \overrightarrow{f^{vol}}, \vec{F}) = \{\sigma : \mathcal{V} \to \mathrm{M}_{3,3}(\mathbf{R}),& \\ \text{such that } \forall \vec{x} \in \mathcal{V},\ \forall i, j \in \{1,2,3\}, \sigma_{ij}(\vec{x}) = \sigma_{ji}(\vec{x}),& \\ \text{and } \forall \vec{x} \in \mathcal{V},\ \overrightarrow{f^{vol}}(\vec{x}) + \frac{\partial \sigma_{ij}(\vec{x})}{\partial x_j}\vec{e_i} = \vec{0},& \\ \text{and } \forall \vec{x} \in \partial_F \mathcal{V},\ \vec{F}(\vec{x}) = \sigma_{ij}(\vec{x}) n_j(\vec{x}) \vec{e_i}\}.&\end{aligned}} \tag{2.50}$$

Tutorial Sheet 4: Some classic statically admissible configurations

Exercise 0. Look for errors in the lecture notes and report your findings.

In this tutorial, we will consider a few symmetric tensor fields in some geometries (filled with an elastic solid) and ask ourselves if they can correspond to statistically admissible stress tensors, and for which system of forces. In all the exercises, one considers a fixed coordinate system with an origin O and an orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. The position of a pont M can therefore be identified with the vector $\overrightarrow{OM} = \vec{x} = x_i\vec{e_i}$. The sum rule over repeated indices is applied.
The questions are very repetitive, you can always ask them given the geometry of a sample of continuous medium and a symmetric tensor field (if the proposed tensor field is not symmetric, do not answer any question, you can declare that it is not statically admissible for any system of forces, because the momenta of the forces would not sum to zero with a non-symmetric stress tensor.

Exercise 1. Uniaxial stress tensor. Consider a continuous medium occupying a cylinder $\mathcal{V}$ with axis parallel to $\vec{e_1}$, radius a and length L. Consider the following stress tensor field in the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$\sigma(\vec{x}) = \begin{pmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

defined for all $\vec{x} \in \mathcal{V}$, where T is a constant (it depends neither or time nor on the point $\vec{x}$). The origin O is chosen to be in the center of one of the disk faces of the cylinder (the center of the other disk has coordinates $(L, 0, 0)$).
0) Draw a figure showing all these geometric data.
1) Express the unit normal vector (pointing towards the exterior) $\vec{n}(x)$ on the boundary $\partial\mathcal{V}$ of the cylinder.
2) Compute the vector field $\sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e_i}$ on $\partial\mathcal{V}$.
3) Compute the vector field $\frac{\partial\sigma_{ij}}{\partial x_j}(\vec{x})\vec{e_i}$ in $\mathcal{V}$.
4) Describe the system of volume forces and surface forces for which the above stress tensor statically admissible.

Exercise 2. Shear stress tensor. Consider a continuous medium occupying a rectangular parallelepiped $\mathcal{V}$ with the axes aligned with the vectors of the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$\mathcal{V} = \left\{\vec{x} = x_i\vec{e_i} \in \mathbf{R}^3, 0 \leq x_1 \leq A, 0 \leq x_2 \leq B, 0 \leq x_3 \leq C\right\}.$$

Consider the following stress tensor field in the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$:

$$\sigma(\vec{x}) = \begin{pmatrix} 0 & F & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

0) Draw a figure showing all these geometric data.
1) Express the unit normal vector (pointing towards the exterior) $\vec{n}(x)$ on the boundary $\partial\mathcal{V}$.
2) Compute the vector field $\sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e_i}$ on $\partial\mathcal{V}$.
3) Compute the vector field $\frac{\partial\sigma_{ij}}{\partial x_j}(\vec{x})\vec{e_i}$ in $\mathcal{V}$.
4) Describe the system of volume forces and surface forces for which the stress tensor 3.3 is statically admissible.

Exercise 3. Torsion stress tensor. Consider a continuous medium occupying a cylinder $\mathcal{V}$ with axis parallel to $\vec{e_3}$, radius a and length L, with the origin at the center of one of the disk faces (the center of the other disk has coordinates $(0,0,L)$). Consider the following stress tensor field in the base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, with entries depending on the coordinates:

$$\sigma(\vec{x}) = \begin{pmatrix} 0 & 0 & -Cx_2 \\ 0 & 0 & Cx_1 \\ -Cx_2 & Cx_1 & 0 \end{pmatrix}$$

0) Draw a figure showing all these geometric data.
1) What is the physical unit of C?
2) Compute the vector field $\sigma_{ij}(\vec{x})n_j(\vec{x})\vec{e_i}$ on $\partial\mathcal{V}$.
3) Compute the vector field $\frac{\partial\sigma_{ij}}{\partial x_j}(\vec{x})\vec{e_i}$ in $\mathcal{V}$.
4) Describe the system of volume forces and surface forces for which this stress tensor field stress tensor is statically admissible.
5) Compute the angular momentum of the forces applied to the top face (of equation $x_3 = L$), at the center of the face.

Exercise 4 (exam question, 2015, for 20 out of 100 marks). Statically admissible stress tensor for an elastic parallelepiped. Consider a parallelepiped of sides $2a, 2b, c$. the space $\mathbf{R}^3$ endowed with an orthonormal base $(\vec{e_1}, \vec{e_2}, \vec{e_3})$. The volume of the parallelepiped is the following set of points, in Cartesian coordinates (x_1, x_2, x_3):

$$\mathcal{V} = \{\vec{x},\ -a \le x_1 \le a,\ -b \le x_2 \le b,\ 0 \le x_3 \le c\}\,.$$

Consider the stress tensor inside $\mathcal{V}$ is given by the following components:

$$\sigma_{12} = \sigma_{21} = Ax_1^2 + Bx_1 + Cx_2 + D,$$

$$\sigma_{23} = \sigma_{32} = Ex_1x_3 + Fx_2,$$

$$\sigma_{33} = Gx_3^2,$$

and

$$\sigma_{ij} = 0 \quad \text{for all other values of } (i,j). \tag{2.51}$$

(i) [**5 marks**] What are the physical units of the constant coefficients A, B, C, D, E, F, G?

(ii) [**10 marks**] Bulk forces are neglected. Write the balance equations $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ for the parallelepiped in terms of the coefficients A, B, C, D, E, F. For what values of these coefficients does σ solve the balance equations?

(iii) [**5 marks**] No external force (**i.e. a zero density of forces**) is applied to the sides of equation $x_1 = \pm a$. Prove that this implies $D = -Aa^2$.

Chapter 3

Linear elasticity

Lecture 5: Linear elasticity

In this lecture we will be interested in small, reversible deformations of solids. We will use Hooke's law to propose a relation between the strain and stress tensors in terms of two parameters: Young's modulus and Poisson's ratio, which are both measurable from traction experiments on elastic solids.

3.1 Hooke's law (for a cylinder)

Traction experiments show that the relative deformation of a cylinder of length L under traction is proportional to the force applied on the ends per unit surface as long as δL is small enough for the transformation to be reversible. This linear relation is known as Hooke's law[1] that has to be applied to the spring to produce this tension. The proportionality coefficient E is known as Young's modulus[2] (or elastic modulus), and has the dimension of a pressure:

$$\boxed{\frac{F}{S} = E\frac{\delta L}{L}.} \tag{3.1}$$

If F is positive one has a *traction*, if it is negative one has a *compression*. For definiteness let us assume that we have a traction. During a traction experiment, the cylinder is going to become thinner, and its diameter can be measured. This effect will be described in the next section by another parameter called Poisson's ratio[3], and denoted by ν. If a denotes the initial radius of the cylinder, and δa the variation of the radius under traction, then ν is the number such that:

$$\boxed{\frac{\delta a}{a} = -\nu\frac{\delta L}{L}.} \tag{3.2}$$

Linear elasticity describes transformation of solids that are sufficiently small to be reversible: if the force F is decreased to zero after a traction experiment *in the linear regime*, the material

[1] The British physicist and polymath Robert Hooke (1635-1703) first established this law in the case of springs (in Latin *ut tension sic vis*, or *such tension, such force*, see Fig. **??**): for a spring δL is proportional to the tension (*tensio*) of the spring, and F is the force (*vis*). Generalizations of this proportionality law to elastic solids, and their formulation in terms of tensor, are still referred to as Hooke's law.

[2] Named after the British physicist Thomas Young (1773-1829), a universal mind who was also a physician and a linguist, and left his name to the Young slits experiment in optics (which was crucial to the development of the study of interferences of light in the 19th century, and again for the interference between electron and atom beams in the 20th century, leading to the development of quantum mechanics).

[3] Named after Siméon Denis Poisson (1781-1840), French mathematician and physicist who also attached his name to the Poisson law in probability

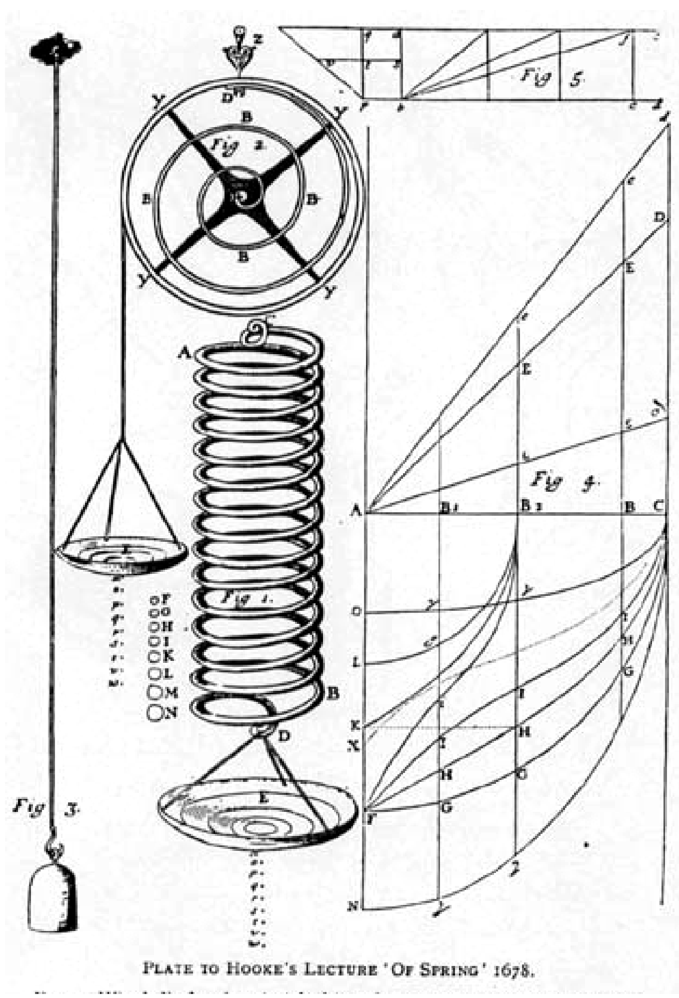

Figure 3.1: A figure from Hooke's original treaty. The applied force is proportional to the masses put in the plate of the scale. In this chapter we are substituting an elastic solid to the spring, and translating this figure into the language of tensors.

comes back to its initial state. The amplitude of the linear regime depends on the material and on the nature of the transformation (for example for steel $\delta L/L$ is of the order of a few thousands in the linear regime, whereas for rubber it can be a few percents), and it has to be determined by experiment, just as Young's modulus. If F is increased beyond the elasticity regime, the is a second regime in which the force can be decreased, while δL decreases without going back to zero (there is a *remanent deformation*). For even larger values of F there is breakage. The scope course of this course is confined to linear elasticity.

Remarks on gravity, linearity and orders of magnitude. In the above discussion we did not specify the direction of the applied forces: we just described them are traction forces, i.e. they are parallel to the axis of the cylinder, i.e. $\vec{F} = F\vec{n}$ at both ends of the cylinder, with $F > 0$ for a traction ($F < 0$ would be a compression), but the normal vector $\vec{n}$ could be horizontal or vertical. This means we neglected gravity (and we will often neglect it in the context of linear elasticity). There are two reasons to neglect gravity:
1. We are interested in small deformations from an initial equilibrium state to a final equilibrium state, and the equilibrium equations in the initial state include gravity and the forces that balance it (for example the reaction of a table if the cylinder rests horizontally on a table, or the reaction of the ceiling if the cylinder is attached to the ceiling). So we begin our work when gravity has already been taken into account, and we look for variations of the stress tensor with respect to this state.
2. For many solids of interest, the value of Young's modulus makes gravity forces smaller than traction forces by orders of magnitude, even for small deformations. In the case of steel, $E \simeq 200\text{GPa}$, and the traction regime is linear for $0 \leq \delta L/L \leq 0.5 \times 10^{-3}$, so if we consider a cylinder of steel with mass $m = 1\text{kg}$ and $S = 10\text{cm}^2$, the effect of gravity by unit of surface is about

$$\frac{mg}{S} = \frac{1 \times 10}{10^{-3}} = 10^4\text{Pa},$$

and that of the traction force per unit surface for $\delta L/L \simeq 0.5 \times 10^{-3}$ is about:

$$\frac{F}{S} = E\frac{\delta L}{L} = 2 \times 10^{11} \times 5 \times 10^{-4} = 10^8\text{Pa},$$

or 10^4 times larger.

3.2 Small deformations and linearized strain tensor

In Lecture 2 we described the kinematics of continuous media in terms of a flow $\vec{\Phi}$ that maps the initial coordinates $X_i\vec{e_i}$ of material points in a continuous medium to a new position at time t, with coordinates $\Phi_i(\vec{X}, t)\vec{e_i}$. In the linear regime where Hooke's law (Eq. 3.1) holds, deformations are small in scale of the original dimensions of the material, so we will express them in terms of the displacement field $\vec{u}$ defined in terms of the flow by subtracting the identical flow:

$$\vec{u}(\vec{X}, t) = u_i(\vec{X}, t)\vec{e_i} = \vec{\Phi}(\vec{X}, t) - \vec{X}, \tag{3.3}$$

meaning in terms of scalar fields:

$$u_i(\vec{X},t) = \Phi_i(\vec{X},t) - X_i, \quad \forall i \in \{1,2,3\} \tag{3.4}$$

so that vector fields (see the chapter on kinematics are transformed as follows by the flow:

$$\vec{h} \mapsto (T_{ij}h_j)\,\vec{e_i}, \tag{3.5}$$

with

$$T_{ij} = \frac{\partial \Phi_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j}. \tag{3.6}$$

We are going to assume that the components u_i for all i are small, and slowly varying over space (i.e. all their derivatives are assumed to be *small*). Let us expand the deformed dot-product of vectors h and h' in powers of u:

$$\boxed{\begin{aligned}\vec{h}.\vec{h'} \mapsto \quad (T_{ij}h_j\vec{e_i}).(T_{lk}h'_k\vec{e_l}) &= h_jh'_kT_{ij}T_{lk}\delta_{il}\\ &= h_jh'_k\left(\delta_{ij}+\tfrac{\partial u_i}{\partial X_j}\right)\left(\delta_{ik}+\tfrac{\partial u_i}{\partial X_k}\right)\\ &= h_jh'_k\left(\delta_{ij}+\tfrac{\partial u_i}{\partial X_j}\right)\left(\delta_{ik}+\tfrac{\partial u_i}{\partial X_k}\right)\\ &= h_jh'_k\left(\delta_{ij}\delta_{ik}+\delta_{ij}\tfrac{\partial u_i}{\partial X_k}+\tfrac{\partial u_i}{\partial X_j}\delta_{ik}+\tfrac{\partial u_i}{\partial X_j}\tfrac{\partial u_i}{\partial X_k}\right)\\ &= h_jh'_k\left(\delta_{jk}+\tfrac{\partial u_j}{\partial X_k}+\tfrac{\partial u_k}{\partial X_j}+\tfrac{\partial u_i}{\partial X_j}\tfrac{\partial u_i}{\partial X_k}\right).\end{aligned}}$$

Hence we can express the first-order terms in $\vec{u}$ (and its derivatives) in terms of the deformation tensor ϵ defined as:

$$\boxed{\epsilon_{ij}(\vec{X},t) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right)}, \tag{3.7}$$

and we find:

$$\boxed{\vec{h}.\vec{h'} \mapsto \vec{h}.\vec{h'} + 2\epsilon_{ij}h_ih'_j + o(\vec{u}).} \tag{3.8}$$

Example: traction of a cylinder. In the special case of the traction on a cylinder, the variation of length δL is proportional to the length, so for a cylinder of initial length αL, with the same base surface S and the same traction F, the length would vary by $\alpha\delta L$. One can express this proportionality rule by writing the displacement field in the direction $\vec{e_3}$ as:

$$u_3 = \frac{X_3}{L}\delta L,$$

where the factor X_3/L plays the role of the factor α. Hence we can express one component of the linearized strain tensor:

$$\epsilon_{33} = \frac{\delta L}{L}.$$

If we look at a section of the cylinder by a plane orthogonal to its axis, we have a disk whose radius is $a = \sqrt{S/\pi}$ when $F = 0$. This radius will be $a + \delta a$ when the traction F is applied, with $\delta a < 0$, and by the same reasoning as for the direction X_3 one can convince oneself that the displacement is linear in both X_1 and X_2 directions:

$$u_1 = \frac{X_1}{a}\delta a,$$

$$u_2 = \frac{X_2}{a}\delta a,$$

so that in particular,

$$\epsilon_{11} = \epsilon_{22} = \frac{\delta a}{a}.$$

Instead of describing this situation in terms of the radius a, which depends on the geometry, one chooses to describe it in terms of the ratio between components of the strain tensor. In terms of Poisson's ratio (the parameter ν defined in Eq. 3.2), for the cylinder in traction, and small deformations, we find:

$$\epsilon_{11} = \epsilon_{22} = -\nu\epsilon_{33}.$$

Poisson's ratio depends on the material, and has no physical dimension. One can compute its value for an incompressible material (exercise, see tutorial), and measure it in traction experiments as it equals the relative variation of the radius (for example $\nu \simeq 0.28$ for steel, and $\nu = 0.2$ for concrete).

3.3 Deformations as a function of stress: a *model* expressing Hooke's law in tensor form

For traction forces acting on the ends of a cylinder, without volume forces, the following uniaxial stress tensor is statically admissible (see Tutorial 4):

$$\sigma(\vec{x}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{F}{S} \end{pmatrix}$$

Hooke's law can therefore be rewritten as the following relation between the entry ϵ_{33} of the tensor ϵ and the entry with the same indices of the stress tensor:

$$\epsilon_{33} = \frac{1}{E}\sigma_{33}. \tag{3.9}$$

But this is not a relation between ϵ and σ as matrices. We can proceed by trial-and-error and ask: *"What if the relation 3.9 held between all the entries of ϵ and σ?"* We would have

$$\epsilon_{ij} \stackrel{?}{=} \frac{1}{E}\sigma_{ij}, \tag{3.10}$$

which is the simplest generalization of Eq. 3.9 to the tensors ϵ and σ. But in that case, the components ϵ_{11} and ϵ_{22} of the deformation tensor would be zero (because σ is uniaxial, Eq. 3.3), therefore the cylinder would not become thinner under traction, which would be in contradiction with our prediction 3.2, as we know Poisson's ratio is measured to be different from zero.

To account for the transverse thinning of the cylinder under traction, we therefore need to add other terms to the r.h.s of 3.9. Linearity and isotropy impose that these terms should have the same eigenvectors as σ (for any choice of σ), and should be a linear function of σ. A natural way to satisfy these two conditions is to add a tensor proportional to the identity matrix, with the trace of σ as a coefficient. We now have two parameters, call them E_1 and E_2:

$$\epsilon_{ij} \stackrel{?}{=} \frac{1}{E_1}\sigma_{ij} + \frac{1}{E_2}(\mathrm{Tr}\sigma)\delta_{ij}. \tag{3.11}$$

Again we ask: *"What if ϵ was given by Eq. 3.11?"*
For the cylinder in pure traction, the l.h.s. of 3.11 equals $F/(ES)$ by Hooke's law and the r.h.s. equals $F/(E_1S) + F/(E_2S)$ from the expression of the uniaxial tensor (Eq. 3.3). In order to recover Hooke's law (Eq. 3.9) for the cylinder in pure traction, we must have

$$\frac{1}{E} = \frac{1}{E_1} + \frac{1}{E_2}. \tag{3.12}$$

Again for the cylinder in pure traction, we can express the diagonal terms of the deformation tensor from Eq. 3.11 as $\epsilon_{11} = F/(E_2S)$ and $\epsilon_{22} = F/(E_2S)$, but from our experimental definition of Poisson's ratio, they must equal $-\nu\epsilon_{33} = -\nu F/(ES)$. Hence the relation

$$-\frac{\nu}{E} = \frac{1}{E_2}. \tag{3.13}$$

By plugging Eq. 3.13 into Eq. 3.12, we can express the parameters E_1 and E_2 in terms of the measurable quantities ν and E, and conclude that the proposed form Eq. 3.11 is compatible with observations in the linear regime of the traction experiment, provided the stress and strain tensors are related by the following equation:

$$\boxed{\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\left(\mathrm{Tr}\sigma\right)\delta_{ij}}. \tag{3.14}$$

We will assume from now on that this educated guess gives us the relation between strain and stress in the linear regime of deformation, for all elastic solids, and not only in the case of pure traction but also for more general systems of applied forces (see tutorials for tests of this idea in the cases of shear deformation and isotropic pressure).

3.4 Stress as a function of deformation

So far we have expressed the deformation tensor ϵ as a function of the stress tensor σ. In order to link dynamics to kinematics we can invert the tensor form of Hooke's law (Eq. 3.14), and obtain the expression of the stress tensor as a function of the deformation tensor.

As Hooke's law can be rewritten as

$$\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{\nu}{1+\nu}\left(\mathrm{Tr}\sigma\right)\delta_{ij}, \tag{3.15}$$

all we need to do is to express $(\mathrm{Tr}\sigma)$ as a function of ϵ Let us take the trace of Eq. 3.14:

$$\mathrm{Tr}\epsilon = \frac{1+\nu}{E}\mathrm{Tr}\sigma - \frac{\nu}{E}\left(\mathrm{Tr}\sigma\right)\delta_{ii}. \tag{3.16}$$

As $\delta_{ii} = 3$, we obtain:

$$\mathrm{Tr}\epsilon = \frac{1-2\nu}{E}\mathrm{Tr}\sigma, \tag{3.17}$$

hence we obtain the following expression for the stress tensor:

$$\boxed{\sigma_{ij} = \frac{E}{1+\nu}\epsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)}\mathrm{Tr}\epsilon\delta_{ij}.} \tag{3.18}$$

Tutorial Sheet 5: Linear elasticity

Exercise 0. Look for errors in the lecture notes and report your findings.

In all the exercises, one considers a fixed coordinate system in $\mathbf{R}^3$,with an origin O and an orthonormal basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$, and the associated Cartesian coordinates (X_1, X_2, X_3). When the time variable t is omitted, this means that a **time large enough for a static equilibrium to be reached** is considered.

Exercise 1. Poisson's ratio for an incompressible elastic solid. Consider again the traction of an elastic cylinder, as in Hooke's law (initial length L, initial radius a without forces initially; a traction F per unit surface is applied of the two disks at both ends, a new equilibrium is reached and the new length is $L + \delta L$, the new radius is $a + \delta a$). **All deformations are supposed to be small, hence you are invited to keep only the terms of order one in δa and δL.**
1) Suppose that the density of the material is uniform, and that the material is incompressible: $\rho(\vec{X}) = \rho$, for all $\vec{X}$, for some constant value ρ in the initial equilibrium, and also in the final equilibrium. Write down the conservation of the mass of the cylinder, and express δa in terms of δL, a and L.
2) Use Hooke's law to express δa and in terms of F, S and Young's modulus E.
What is the value of ν for an incompressible material?

Exercise 2. Variation of volume under small deformations. Write the transformation under a flow $\vec{X} \mapsto X_i\vec{e_i} + \vec{u}(\vec{X})$ of the three basis vectors $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$. Compute the determinant of the system of three vectors before and after the transformation. How does the volume of an infinitesimal cube transform as a function of u (working at lowest non-zero order in u and all its derivatives)?

Lecture 6: Linear elasticity and application to spherical shells

So far we have introduced Hooke's law as a material law for elastic solids, but we have not worked out the system of PDEs that allows to work out the displacement field in small deformations (denoted by $\vec{u}$). We happened to be able to calculate $\vec{u}$ in the very simple case of an elastic cylinder under traction. In more complicated geometries and force configurations, we can obtain equations for the displecement field by injecting the material law into the balance equations. In this lecture we will write down these equations, called the Navier equations, and study them with boundary conditions in a spherical geometry. Throughout these notes, $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ denotes an orthonormal base of $\mathbf{R}^3$, and the position $\vec{x}$ of a point in space is described through associated Cartesian coordinates defined by $\vec{x} = x_i \vec{e_i}$.

3.5 The Navier equations

We start with the balance equations

$$\vec{0} = \overrightarrow{f^{vol}} + \frac{\partial \sigma_{ij}}{\partial x_j} \vec{e_i}, \tag{3.19}$$

where $\overrightarrow{f^{vol}} = f_i^{vol} \vec{e_i}$ is the density of volume forces, and the material law relating the stress tensor to the deformation field $\vec{u} = u_i \vec{e_i}$, in the regime of linear elasticity, i.e. when all components of $\vec{u}$ are very small.

$$\epsilon_{ij} = \frac{1+\nu}{E} \sigma_{ij} - \frac{\nu}{E} (\mathrm{Tr}\sigma) \delta_{ij}, \tag{3.20}$$

where ϵ denotes the linearized strain tensor:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right). \tag{3.21}$$

We would like to express Eq. 4.1 in terms of the field $\vec{u}$, so the first thing we have to do is to 'invert' Hooke's law in order to express σ as a function of ϵ. Since Hooke's law is linear, this is easily done by taking the trace of both sides of Eq. 4.3:

$$\mathrm{Tr}\epsilon = \frac{1+\nu}{E} \mathrm{Tr}\sigma - 3\frac{\nu}{E} \mathrm{Tr}\sigma = \frac{1-2\nu}{E} \mathrm{Tr}\sigma, \tag{3.22}$$

hence Hooke's law becomes

$$\sigma_{ij} = \frac{E}{1+\nu} \left(\epsilon_{ij} + \frac{\nu}{E} \mathrm{Tr}\sigma \delta_{ij} \right) = \frac{E}{1+\nu} \epsilon_{ij} + \frac{E\nu}{(1-2\nu)(1+\nu)} (\mathrm{Tr}\epsilon) \delta_{ij}. \tag{3.23}$$

It is usual to rewrite this equation in terms of the Lamé coefficients μ and λ defined (note the factor of 2) as

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda(\mathrm{Tr}\epsilon)\delta_{ij}, \tag{3.24}$$

from which we read off

$$\boxed{\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.} \tag{3.25}$$

To describe the elastic properties of a solid, one can either choose Young's modulus and Poisson's ratio or the Lamé coefficients. Navier's equation are often written using the Lamé coefficients. Substituting the form of the material law written in Eq. 4.6, we obtain **for all** i **in** [1..3]:

$$\boxed{0 = f_i^{vol} + \mu\,\triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right),} \tag{3.26}$$

which are called the Navier equations of linear elasticity. They are a system of three scalar PDEs, like the balance equations, but the unknowns are the components of the displacement field $\vec{u}$. Note that the stress tensor σ does not appear explicitly in the Navier equations.

3.6 Solution in the case of a spherical shell

3.6.1 Explicit form of the Navier equations (with spherical symmetry)

Consider an elastic spherical shell (meaning the region between concentric spheres of radius R_1 and R_2, with $R_1 \leq R_2$, is filled with an elastic material). A uniform pressure P_2 is applied on the outside, a uniform pressure P_1 is applied on the inside. Volume forces are neglected (or rather they have been compensated by the reaction of some support in the reference configuration). The spherical shell contracts under the influence of the pressure. Let us compute the displacement fields.

• **Use spherical symmetry.** As the problem is spherically symmetric (because of the spherical geometry and the uniform pressure), the displacement field will also have spherical symmetry, so we look for solutions of the form

$$\vec{u}(x_1, x_2, x_3) = \phi(\sqrt{x_i x_i})\vec{x}, \tag{3.27}$$

where we expressed the distance to the origin in terms of Cartesian coordinates, in order to avoid having to look up the expression of differential operators in spherical coordinates. All we have to do is therefor to compute the function ϕ.

• **Work out all terms in the Navier equations in terms of the unknown function** ϕ**.** We have to solve the Navier equations

$$0 = \mu\,\triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right). \tag{3.28}$$

To that end, let us rewrite them as a set of three differential equations in the function ϕ. We will need the expression of the derivatives of the displacement fields:

$$\frac{\partial u_i}{\partial x_j}(x_1,x_2,x_3) = \delta_{ij}\phi(\sqrt{x_k x_k}) + x_i\frac{\partial}{\partial x_j}(\phi(\sqrt{x_k x_k})) = \delta_{ij}\phi(\sqrt{x_k x_k}) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}), \quad (3.29)$$

from which we can compute the divergence of the displacement field needed in the Navier equations:

$$\frac{\partial u_k}{\partial x_k}(x_1,x_2,x_3) = 3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}). \quad (3.30)$$

The second term in Navier's equation is obtained by taking one more derivative:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)(x_1,x_2,x_3) \quad (3.31)$$

$$= \frac{\partial}{\partial x_i}\left(3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k})\right) \quad (3.32)$$

$$= 3\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\frac{x_i}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \quad (3.33)$$

$$= 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \quad (3.34)$$

On the other hand, we can express the Laplacian term as follows:

$$\triangle u_i = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)(x_1,x_2,x_3) = \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{3x_i + \delta_{ij}x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i x_j\frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k})\right) \quad (3.35)$$

but

$$x_i x_j\frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k})\right) = -x_i x_j\frac{x_j}{(x_k x_k)^{\frac{3}{2}}}\phi'(\sqrt{x_k x_k}) + x_i x_j\frac{1}{\sqrt{x_k x_k}}\frac{x_j}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \quad (3.36)$$

$$= -\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \quad (3.37)$$

Putting all the terms together we find exactly the same form for the Laplacian term as in Eq. 4.16:

$$\triangle u_i = 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}), \quad (3.38)$$

hence the Navier equations take the form:

$$0 = (\lambda + 2\mu)x_i\left(4\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \phi''(\sqrt{x_k x_k})\right), \quad (3.39)$$

for all i in [1..3]. These three equations are identical (up to a multiplication by x_i so we obtain the following scalar differential equation for ϕ:

$$0 = \frac{4}{r}\phi'(r) + \phi''(r). \quad (3.40)$$

We can look for solutions of the forl $\phi' = Ar^{\alpha} + B$, and obtain $\alpha = -4$ and $B = 0$. One more integration gives rise to

$$\boxed{\phi(r) = -\frac{A}{3r^3} + C.} \tag{3.41}$$

3.6.2 Determination of the integration constants using boundary conditions

We have to express the boundary conditions on the internal and external spheres using Hooke's law. From our computations of derivatives of displacements (Eq. 4.11) we obtain:

$$\epsilon_{ij} = \delta_{ij}\phi(x_1, x_2, x_3) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}). \tag{3.42}$$

whose trace gives us

$$\mathrm{Tr}(\epsilon) = 3\phi + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}), \tag{3.43}$$

which we can express in sperical coordinates as

$$\mathrm{Tr}(\epsilon) = 3\phi + r\phi'(r) = 3C \tag{3.44}$$

Consider the following two points, one on the internal sphere $\vec{a} = R_1\vec{e_1}$, and one on the outer sphere, $\vec{a} = R_2\vec{e_1}$. The outward-pointing unit normal vector to the shell is $-\vec{e_1}$ at $\vec{a}$ and $+\vec{e_1}$ at $\vec{b}$, hence

$$\sigma_{ij}n_j\vec{e_i} = +P_1\vec{e_1} \text{ on the inside,} \tag{3.45}$$

and

$$\sigma_{ij}n_j\vec{e_i} = -P_2\vec{e_1} \text{ on the outside.} \tag{3.46}$$

Using Hooke's law we therefore obtain (for components of indices $i = j = 1$):

$$-P_1 = 2\mu\left(-2\frac{A}{3R_1^3} + C\right) + 3\lambda C, \tag{3.47}$$

$$-P_2 = 2\mu\left(-2\frac{A}{3R_2^3} + C\right) + 3\lambda C. \tag{3.48}$$

from which one obtains the constants:

$$\boxed{A = \frac{3(P_1 - P_2)R_1^3R_2^3}{4\mu(R_2^3 - R_1^3)},} \tag{3.49}$$

$$\boxed{C = \frac{1}{3\lambda + 2\mu}\frac{P_1R_1^3 - P_2R_2^3}{R_2^3 - R_1^3}.} \tag{3.50}$$

Tutorial Sheet 6: Linear elasticity, the Navier equation

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. Balance equations in a *cylindrical* shell under *lateral* compression. Consider an elastic solid obeying Hooke's law:

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}(\mathrm{Tr}\sigma)\delta_{ij}. \tag{3.51}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{3.52}$$

is the linearised strain tensor, σ is the stress tensor, E is Young's modulus and ν is Poisson's ratio for the material. Deformations $\vec{u} = u_i\vec{e_i}$ are assumed to be small enough for models of linear elasticity to be applicable.
Let $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal base. Consider a cylindrical shell (a tube) of axis $\vec{e_3}$ of internal radius R_1 and external radius R_2, made of this material.
A uniform pressure P_1 is applied to the internal surface of the cylinder. A uniform pressure P_2 is applied to the lateral external surface of the cylinder, and the extremities of the cylindrical shell are fixed, meaning $u_3 = 0$ in the shell.
We want to compute u_1 and u_2 in order to determine the variation of the radius induced by these external forces (lateral compression).

(*i*) Because of cylindrical symmetry, one can write the deformation $\vec{u}$ using a scalar function f depending only on the combination $\sqrt{x_1^2 + x_2^2}$:

$$\vec{u}(\vec{x}) = x_1 f(\sqrt{x_1^2 + x_2^2})\vec{e_1} + x_2 f(\sqrt{x_1^2 + x_2^2})\vec{e_2}. \tag{3.53}$$

Compute $\frac{\partial u_i}{\partial x_j}$ for all i and j in [1..3].

(*ii*) The Lamé coefficients μ and λ are defined as follows:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda(\mathrm{Tr}\epsilon)\delta_{ij}. \tag{3.54}$$

What is the expression of μ and λ in terms of E and ν?

(*iii*) Volume forces are neglected. Show that the balance equations for the cylinder can be written as follows in terms of $\vec{u}$:

$$0 = \mu \triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}(\mathrm{div}\,\vec{u}). \tag{3.55}$$

(iv) Show that Eq. 4.37 implies a differential equation for f of the form

$$\frac{k}{r} f'(r) + f''(r) = 0. \tag{3.56}$$

What is the value of k?

(v) Write down the boundary conditions and use them to integrate the differential equation, introducing (and calculating) as many integration constants as you need.

Chapter 4

Viscous (incompressible, Newtonian) fluids

Lecture 6: Linear elasticity and application to spherical shells

So far we have introduced Hooke's law as a material law for elastic solids, but we have not worked out the system of PDEs that allows to work out the displacement field in small deformations (denoted by $\vec{u}$). We happened to be able to calculate $\vec{u}$ in the very simple case of an elastic cylinder under traction. In more complicated geometries and force configurations, we can obtain equations for the displecement field by injecting the material law into the balance equations. In this lecture we will write down these equations, called the Navier equations, and study them with boundary conditions in a spherical geometry. Throughout these notes, $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ denotes an orthonormal base of $\mathbf{R}^3$, and the position $\vec{x}$ of a point in space is described through associated Cartesian coordinates defined by $\vec{x} = x_i \vec{e_i}$.

4.1 The Navier equations

We start with the balance equations

$$\vec{0} = \overrightarrow{f^{vol}} + \frac{\partial \sigma_{ij}}{\partial x_j} \vec{e_i}, \tag{4.1}$$

where $\overrightarrow{f^{vol}} = f_i^{vol} \vec{e_i}$ is the density of volume forces, and the material law relating the stress tensor to the deformation field $\vec{u} = u_i \vec{e_i}$, in the regime of linear elasticity, i.e. when all components of $\vec{u}$ are very small.

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}(\mathrm{Tr}\sigma)\delta_{ij}, \tag{4.2}$$

where ϵ denotes the linearized strain tensor:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4.3}$$

We would like to express Eq. 4.1 in terms of the field $\vec{u}$, so the first thing we have to do is to 'invert' Hooke's law in order to express σ as a function of ϵ. Since Hooke's law is linear, this is easily done by taking the trace of both sides of Eq. 4.3:

$$\mathrm{Tr}\epsilon = \frac{1+\nu}{E}\mathrm{Tr}\sigma - 3\frac{\nu}{E}\mathrm{Tr}\sigma = \frac{1-2\nu}{E}\mathrm{Tr}\sigma, \tag{4.4}$$

hence Hooke's law becomes

$$\sigma_{ij} = \frac{E}{1+\nu}\left(\epsilon_{ij} + \frac{\nu}{E}\mathrm{Tr}\sigma\delta_{ij}\right) = \frac{E}{1+\nu}\epsilon_{ij} + \frac{E\nu}{(1-2\nu)(1+\nu)}(\mathrm{Tr}\epsilon)\delta_{ij}. \tag{4.5}$$

It is usual to rewrite this equation in terms of the Lamé coefficients μ and λ defined (note the factor of 2) as

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda(\mathrm{Tr}\epsilon)\delta_{ij}, \tag{4.6}$$

from which we read off

$$\boxed{\mu = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.} \tag{4.7}$$

To describe the elastic properties of a solid, one can either choose Young's modulus and Poisson's ratio or the Lamé coefficients. Navier's equation are often written using the Lamé coefficients. Substituting the form of the material law written in Eq. 4.6, we obtain **for all** i **in** [1..3]:

$$\boxed{0 = f_i^{vol} + \mu \triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right),} \tag{4.8}$$

which are called the Navier equations of linear elasticity. They are a system of three scalar PDEs, like the balance equations, but the unknowns are the components of the displacement field $\vec{u}$. Note that the stress tensor σ does not appear explicitly in the Navier equations.

4.2 Solution in the case of a spherical shell

4.2.1 Explicit form of the Navier equations (with spherical symmetry)

Consider an elastic spherical shell (meaning the region between concentric spheres of radius R_1 and R_2, with $R_1 \leq R_2$, is filled with an elastic material). A uniform pressure P_2 is applied on the outside, a uniform pressure P_1 is applied on the inside. Volume forces are neglected (or rather they have been compensated by the reaction of some support in the reference configuration). The spherical shell contracts under the influence of the pressure. Let us compute the displacement fields.

- **Use spherical symmetry.** As the problem is spherically symmetric (because of the spherical geometry and the uniform pressure), the displacement field will also have spherical symmetry, so we look for solutions of the form

$$\vec{u}(x_1, x_2, x_3) = \phi(\sqrt{x_i x_i})\vec{x}, \tag{4.9}$$

where we expressed the distance to the origin in terms of Cartesian coordinates, in order to avoid having to look up the expression of differential operators in spherical coordinates. All we have to do is therefor to compute the function ϕ.

- **Work out all terms in the Navier equations in terms of the unknown function** ϕ**.** We have to solve the Navier equations

$$0 = \mu \triangle u_i + (\lambda+\mu)\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right). \tag{4.10}$$

To that end, let us rewrite them as a set of three differential equations in the function ϕ. We will need the expression of the derivatives of the displacement fields:

$$\frac{\partial u_i}{\partial x_j}(x_1, x_2, x_3) = \delta_{ij}\phi(\sqrt{x_k x_k}) + x_i \frac{\partial}{\partial x_j}(\phi(\sqrt{x_k x_k})) = \delta_{ij}\phi(\sqrt{x_k x_k}) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}), \tag{4.11}$$

from which we can compute the divergence of the displacement field needed in the Navier equations:

$$\frac{\partial u_k}{\partial x_k}(x_1, x_2, x_3) = 3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}). \tag{4.12}$$

The second term in Navier's equation is obtained by taking one more derivative:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial u_k}{\partial x_k}\right)(x_1, x_2, x_3) \tag{4.13}$$

$$= \frac{\partial}{\partial x_i}\left(3\phi(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k})\right) \tag{4.14}$$

$$= 3\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \sqrt{x_k x_k}\frac{x_i}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \tag{4.15}$$

$$= 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \tag{4.16}$$

On the other hand, we can express the Laplacian term as follows:

$$\triangle u_i = \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)(x_1, x_2, x_3) = \frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \frac{3x_i + \delta_{ij}x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i x_j \frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k})\right) \tag{4.17}$$

but

$$x_i x_j \frac{\partial}{\partial x_j}\left(\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k})\right) = -x_i x_j \frac{x_j}{(x_k x_k)^{\frac{3}{2}}}\phi'(\sqrt{x_k x_k}) + x_i x_j \frac{1}{\sqrt{x_k x_k}}\frac{x_j}{\sqrt{x_k x_k}}\phi''(\sqrt{x_k x_k}) \tag{4.18}$$

$$= -\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}). \tag{4.19}$$

Putting all the terms together we find exactly the same form for the Laplacian term as in Eq. 4.16:

$$\triangle u_i = 4\frac{x_i}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + x_i\phi''(\sqrt{x_k x_k}), \tag{4.20}$$

hence the Navier equations take the form:

$$0 = (\lambda + 2\mu)x_i\left(4\frac{1}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}) + \phi''(\sqrt{x_k x_k})\right), \tag{4.21}$$

for all i in [1..3]. These three equations are identical (up to a multiplication by x_i so we obtain the following scalar differential equation for ϕ:

$$0 = \frac{4}{r}\phi'(r) + \phi''(r). \tag{4.22}$$

We can look for solutions of the forl $\phi' = Ar^{\alpha} + B$, and obtain $\alpha = -4$ and $B = 0$. One more integration gives rise to

$$\boxed{\phi(r) = -\frac{A}{3r^3} + C.} \tag{4.23}$$

4.2.2 Determination of the integration constants using boundary conditions

We have to express the boundary conditions on the internal and external spheres using Hooke's law. From our computations of derivatives of displacements (Eq. 4.11) we obtain:

$$\epsilon_{ij} = \delta_{ij}\phi(x_1, x_2, x_3) + \frac{x_i x_j}{\sqrt{x_k x_k}}\phi'(\sqrt{x_k x_k}). \tag{4.24}$$

whose trace gives us

$$\mathrm{Tr}(\epsilon) = 3\phi + \sqrt{x_k x_k}\phi'(\sqrt{x_k x_k}), \tag{4.25}$$

which we can express in sperical coordinates as

$$\mathrm{Tr}(\epsilon) = 3\phi + r\phi'(r) = 3C \tag{4.26}$$

Consider the following two points, one on the internal sphere $\vec{a} = R_1\vec{e_1}$, and one on the outer sphere, $\vec{a} = R_2\vec{e_1}$. The outward-pointing unit normal vector to the shell is $-\vec{e_1}$ at $\vec{a}$ and $+\vec{e_1}$ at $\vec{b}$, hence

$$\sigma_{ij}n_j\vec{e_i} = +P_1\vec{e_1} \text{ on the inside,} \tag{4.27}$$

and

$$\sigma_{ij}n_j\vec{e_i} = -P_2\vec{e_1} \text{ on the outside.} \tag{4.28}$$

Using Hooke's law we therefore obtain (for components of indices $i = j = 1$):

$$-P_1 = 2\mu\left(-2\frac{A}{3R_1^3} + C\right) + 3\lambda C, \tag{4.29}$$

$$-P_2 = 2\mu\left(-2\frac{A}{3R_2^3} + C\right) + 3\lambda C. \tag{4.30}$$

from which one obtains the constants:

$$\boxed{A = \frac{3(P_1 - P_2)R_1^3 R_2^3}{4\mu(R_2^3 - R_1^3)},} \tag{4.31}$$

$$\boxed{C = \frac{1}{3\lambda + 2\mu}\frac{P_1 R_1^3 - P_2 R_2^3}{R_2^3 - R_1^3}.} \tag{4.32}$$

Tutorial Sheet 6: Linear elasticity, the Navier equation

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. Balance equations in a *cylindrical* shell under *lateral* compression. Consider an elastic solid obeying Hooke's law:

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}(\mathrm{Tr}\sigma)\delta_{ij}. \tag{4.33}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{4.34}$$

is the linearised strain tensor, σ is the stress tensor, E is Young's modulus and ν is Poisson's ratio for the material. Deformations $\vec{u} = u_i\vec{e_i}$ are assumed to be small enough for models of linear elasticity to be applicable.
Let $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ be an orthonormal base. Consider a cylindrical shell (a tube) of axis $\vec{e_3}$ of internal radius R_1 and external radius R_2, made of this material.
A uniform pressure P_1 is applied to the internal surface of the cylinder. A uniform pressure P_2 is applied to the lateral external surface of the cylinder, and the extremities of the cylindrical shell are fixed, meaning $u_3 = 0$ in the shell.
We want to compute u_1 and u_2 in order to determine the variation of the radius induced by these external forces (lateral compression).

(i) Because of cylindrical symmetry, one can write the deformation $\vec{u}$ using a scalar function f depending only on the combination $\sqrt{x_1^2 + x_2^2}$:

$$\vec{u}(\vec{x}) = x_1 f(\sqrt{x_1^2 + x_2^2})\vec{e_1} + x_2 f(\sqrt{x_1^2 + x_2^2})\vec{e_2}. \tag{4.35}$$

Compute $\frac{\partial u_i}{\partial x_j}$ for all i and j in [1..3].

(ii) The Lamé coefficients μ and λ are defined as follows:

$$\sigma_{ij} = 2\mu\epsilon_{ij} + \lambda(\mathrm{Tr}\epsilon)\delta_{ij}. \tag{4.36}$$

What is the expression of μ and λ in terms of E and ν?

(iii) Volume forces are neglected. Show that the balance equations for the cylinder can be written as follows in terms of $\vec{u}$:

$$0 = \mu \triangle u_i + (\lambda + \mu)\frac{\partial}{\partial x_i}(\mathrm{div}\,\vec{u}). \tag{4.37}$$

(iv) Show that Eq. 4.37 implies a differential equation for f of the form

$$\frac{k}{r} f'(r) + f''(r) = 0. \tag{4.38}$$

What is the value of k?

(v) Write down the boundary conditions and use them to integrate the differential equation, introducing (and calculating) as many integration constants as you need.

Lecture 7: Viscous fluids

Keywords. Description of fluids, conservation of mass, material laws, Newtonian fluids.

4.3 Summary of the module so far

We have developed a general framework to study continuous media.
(i) Continuum assumption: distances are large in scale of the size of molecules.
(ii) Kinematics: Lagrangian description based on trajectories $\vec{\Phi}(\vec{X},t)$, Eulerian description based on a velocity field $\vec{v}(\vec{x},t)$. The two descriptions are related by $\vec{v}(\vec{\Phi}(\vec{X},t),t) = \frac{\partial}{\partial t}\vec{\Phi}(\vec{X},t)$.
(iii) bulk forces and surface forces. Cauchy's assumption: consider a sample of continuous medium in a volume $\mathcal{V}$, whose boundary $\mathcal{S}$ is oriented towards the exterior of $\mathcal{V}$, this sample feels the action of the rest of the continuous medium through the force $\left(\oiint \sigma_{ij}(\vec{x})n_j(\vec{x})dS\right)\vec{e_i}$, where σ is the stress tensor and $\vec{n}(\vec{x}) = n_j(\vec{x})\vec{e_j}$ is the normal vector to $\mathcal{S}$ at point $\vec{x}$, and $(\vec{e_1},\vec{e_2},\vec{e_3})$ is an orthonormal base.

We specialized this framework for the study of elastic solids (in the regime of small deformations) and we have started doing it for fluids.

• **Elastic solids.**
(i) We study equilibrium states. Thanks to Stoke's theorem the balance laws can be written as follows for all i in [1..3]:

$$\vec{0} = \overrightarrow{f^{vol}} + \frac{\partial \sigma_{ij}}{\partial x_j}\vec{e_i} \tag{4.39}$$

(ii) If forces are small enough, deformations are small and reversible. Computations are done at first order in the deformation field $\vec{u} = \Phi(\vec{X},t) - \vec{X}$.
(iii) The stress tensor in terms of the linearized strain tensor ϵ by Hooke's law, expressed using two parameters, Young's modulus E (order of magnitude: a few hundreds of gigapascals for steel, about 10 gigapascals for wood, about 0.1 gigapascal for rubber), and Poisson's ratio ν (a positive number which can be shown to be smaller than 0.5):

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}(\mathrm{Tr}\sigma)\delta_{ij} \qquad \epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{4.40}$$

• **Fluids.**
(i) We study flows, deformations are large, hence we need to include the acceleration terms in

the equations of motion:

$$\rho\left(\frac{\partial \vec{v}}{\partial t}+(\vec{v}.\vec{\nabla})\vec{v}\right)=\overrightarrow{f^{vol}}+\frac{\partial \sigma_{ij}}{\partial x_j}\vec{e_i}. \tag{4.41}$$

(*ii*) We need a *material law* (analogous to Hooke's law) to express the stress tensor σ in terms of the other parameters of the problem.

4.4 Reminders on fluids

Fluids are continuous media that can undergo large deformation without losing their continuity property: for example a volume of water can be transferred between two containers of different shapes, and it will still be a single volume of water (provided the pressure and temperature conditions are sufficiently similar in the two containers), whereas for elastic solids the identity of the material can be modified even by relatively small deformation (for a bar of steel, the linear regime of elasticity is only valid for $\delta L/L \simeq 5.10^{-3}$, and breakage, i.e. loss of continuity, can occur if $\delta L/L$ reaches a few percent).

This will have several consequences for us:
1. We will need a description of speed and position that easily accommodates $\vec{\Phi}(\vec{X},t)=\vec{X}+\vec{u}(\vec{X},t)$ where $||\vec{u}||$ is not necesseraly small in scale of $||\vec{X}||$; this description is called the Euler description;
2. As we will study flows, not only hydrostatics, we will need to add an acceleration term to the equations of motion.

4.4.1 The Euler description of fluids

The Euler velocity field is a vector field (it associates a vector to every point in space, at every time), denoted by $\vec{v}(\vec{x},t)$. Its value is the speed of the fluid particle that is at point $\vec{x}$ at time t. The relation to the Lagrange description is the following: the particle that is at point $\vec{x}$ at time t was at some point $\vec{X}$ at time 0, so for this particle one has $\vec{x}=\vec{\Phi}(\vec{x},t)$. its speed at time t is just the derivative of the position $\vec{\Phi}(\vec{x},t)$ with respect to time, hence one can compute the Euler velocity field in terms of the Lagrange flow function $\vec{\Phi}$ as follows:

$$\boxed{\vec{v}(\vec{\Phi}(\vec{X},t),t)=\frac{\partial}{\partial t}\vec{\Phi}(\vec{X},t).}$$

In the case of the flow of a river, or inside a pipe, we will write the equations of motion in terms of the Euler velocity field $\vec{v}$ and its derivative, as it is more practical to measure the velocity of the fluid that is passing in front of the observer (at point $\vec{x}$) than to ask oneself where each particle was at time 0.

Remark ('What is the Euler description for elastic solids ?'). In the case of elastic solids we consider small deformations. We computed all the effects at the lowest non-zero orders in the deformation (and their derivatives) and only used the Lagrange description. Let us see what the relation between the two descriptions becomes for small deformation:

$$\vec{\Phi}(\vec{X},t) = \vec{X} + \vec{u}(\vec{X},t), \tag{4.42}$$

hence the relation between Euler and Lagrange descriptions becomes

$$\vec{v}(\vec{X} + \vec{u}(\vec{X},t),t) = \frac{\partial}{\partial t}\vec{u}(\vec{X},t), \tag{4.43}$$

from which we see that the velocity is of order one in u, hence if we are working at lowest non-zero order in u (and its derivatives), we can replace the argument $\vec{X} + \vec{u}$ by u, and the Eulerian field is just the time derivative of the Lagrange flow, expressed at the same point:

$$\vec{v}(\vec{X},t) = \frac{\partial}{\partial t}\vec{u}(\vec{X},t) + o(\vec{u}), \tag{4.44}$$

hence the Lagrange and Euler descriptions are 'trivially' equivalent for small deformations.

The Euler description will be used to write the equation of motions for fluids. However, the Lagrange viewpoint (in which one follows the same particles of fluids along their trajectory) can be useful when establishing the equations, as we are going to see in the next two sections.

4.4.2 Conservation of mass ("the continuity equation")

Consider the fluid enclosed in volume $\mathcal{V}$ at time t. We can write its mass M as

$$M = \iiint_{\mathcal{V}} \rho(\vec{x},t)dV. \tag{4.45}$$

where ρ is the density of the fluid, which *a priori* depends on both space and time. If we consider the same particles of fluid at time $t+\epsilon$ (for some small time ϵ), its mass will still be M by conservation of matter, but its mathematical expression (including terms of order up to one in ϵ) will consist of two terms: one is the volume integral of the density $\rho(\vec{x},t+\epsilon)$ over volume V, and the other one is the surface integral corresponding to the mass particles of fluids that went through the surface between time t and time $t+\epsilon$ (if a particle goes through the boundary $\mathcal{S}$ at point $\vec{x}$, with the outgoing unit normal vector $\vec{n}(\vec{x})$ at this point, the scalar product $\vec{v}(\vec{x},t).\vec{n}(\vec{x})$ is positive if the particle is leaving the volume $\mathcal{V}$, and we must add its mass $\vec{v}(\vec{x},t).\vec{n}(\vec{x})\epsilon dS$ to find M; if the dot-product is negative, the term $\vec{v}(\vec{x},t).\vec{n}(\vec{x})\epsilon dS$ is negative and equals the opposite mass of a fluid particle that was not in volume $\mathcal{V}$ at time t but came in between t and $t+\epsilon$). Hence we write the new expression of M at time $t+\epsilon$ as

$$\begin{aligned} M &= \iiint_{\mathcal{V}} \rho(\vec{x},t+\epsilon)dV + \oiint_{\mathcal{S}\to\text{ext}} \rho(\vec{x},t)\vec{v}(\vec{x},t).\vec{n}(\vec{x})\epsilon dS + o(\epsilon) \\ &= \iiint_{\mathcal{V}} \rho(\vec{x},t)dV + \iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}(\vec{x},t)\epsilon dV + \epsilon \iiint_{\mathcal{V}} \frac{\partial(\rho\vec{v})}{\partial x_j}(\vec{x},t)dV + o(\epsilon) \\ &= M + \epsilon\left(\iiint_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}(\vec{x},t) + \frac{\partial(\rho\vec{v})}{\partial x_j}(\vec{x},t)\right)dV\right) + o(\epsilon) \end{aligned}$$

where we used Stoke's theorem in the second equality. We can write the following equality, which expresses that the derivative of the mass m with respect to ϵ is zero:

$$\iiint_{\mathcal{V}} \left(\frac{\partial \rho}{\partial t}(\vec{x}, t) + \frac{\partial (\rho \vec{v})}{\partial x_j}(\vec{x}, t) \right) dV = 0. \tag{4.46}$$

Since Eq. 4.46 holds for any volume $\mathcal{V}$, it must hold locally (at scales where the continuous description holds), hence the conservation of mass is express as the following PDE in the density and velocity fields:

$$\boxed{\frac{\partial \rho}{\partial t}(\vec{x}, t) + \frac{\partial (\rho v_j)}{\partial x_j}(\vec{x}, t) = 0.} \tag{4.47}$$

This equation is sometimes called the continuity equation (but the mathematical assumptions on functions are stronger than continuity, as we assume that derivatives exist; in this context, *continuity* refers to *conservation* of a quantity, and express that *the amount of a conserved quantity that enters a domain equals the amount that is stored*).

Example (incompressible fluid). In the case where one assumes that the density is uniform and constant (i.e. that the fluid is incompressible), one can write $\rho(\vec{x}, t) = \rho_0$, for example $\rho_0 = 10^3 \text{kg.m}^{-3}$ for water, then Eq. 4.47 becomes and equation in the velocity field only:

$$\frac{\partial v_j}{\partial x_j} = 0. \tag{4.48}$$

which is often written as $\text{div}\vec{v} = 0$, introducing the divergence operator. Hence, given an expression for the velocity fluid of a fluid, one can decide if the fluid is incompressible just by computing the divergence of the velocity field.

4.4.3 Acceleration of a particle of fluid

So far we have studied elasticity problems and wrote that the sum of volume forces and surface forces equals zero. The resulting equation applies to static problems. When we want to write down the dynamics of a flow, the sum of forces takes the same form, but it equals the variation rate of the impulsion. We therefore have to compute the variation of the impulsion of the impulsion of the quantity of fluid contained in volume dV at time t, that is:

$$\vec{p} = \rho(\vec{x}, t) dV \vec{v}(\vec{x}, t), \tag{4.49}$$

between time t and time $t + \epsilon$, at first order in ϵ. The quantity $\vec{p}$ in Eq. 4.49 is expressed in Eulerian variables, but we are following the same particle of fluid on its trajectory between times t and $t + \epsilon$ (which is a Lagrangian approach to the problem, but is still tractable if ϵ is small enough). First of all, the mass of the particle $\vec{\rho}(\vec{x}, t) dV$ is conserved between times t and $t + \epsilon$ (the density may vary, but its variation is compensated by a variation of the volume element dV), so all we have to do is to compute the variation $\delta\vec{v}$ of the velocity of the particle, at first order in ϵ:

$$\delta\vec{v} = \vec{v}(\vec{x} + \vec{v}(\vec{x}, t)\epsilon + o(\epsilon), t + \epsilon) - \vec{v}(\vec{x}, t). \tag{4.50}$$

The delicate point comes from the fact that *both* position and time vary during the move. We can expand the first term in Eq. 4.52 at first order in ϵ, and both time and space derivatives of the Eulerian velocity appear:

$$\vec{v}(\vec{x} + \vec{v}(\vec{x}, t)\epsilon + o(\epsilon), t + \epsilon) = \vec{v}(\vec{x}, t) + v_i(\vec{x}, t)\epsilon \frac{\partial}{\partial x_i}\vec{v}(\vec{x}, t) + \epsilon \frac{\partial}{\partial t}\vec{v}(\vec{x}, t)\epsilon. \tag{4.51}$$

Hence

$$\delta\vec{v} = \epsilon \left(v_i \frac{\partial}{\partial x_i}\vec{v} + \frac{\partial}{\partial t}\vec{v}(\vec{x}, t) \right) + o(\epsilon), \tag{4.52}$$

where all velocities in the r.h.s are now taken at point $\vec{x}$ and time t. The quantity we have just computed is often called the *particle derivative* of the velocity (with a differential symbol D instead of d or ∂), because it is obtained by following the same particle of fluid along its trajectory:

$$\frac{D\vec{v}}{Dt} = \frac{\partial}{\partial t}\vec{v} + v_i \frac{\partial}{\partial x_i}\vec{v}. \tag{4.53}$$

Note that it is different from the time derivative of the velocity, and that the difference is non-linear in the velocity field $\vec{v}$: the corresponding term is often called the convection term. By Cauchy's assumption and Stokes' theorem, the sum of forces acting our the fluid particle is:

$$d\vec{F} = f_i^{vol}\vec{e_i}dV + \frac{\partial}{\partial x_j}\sigma_{ij}\vec{e_i}dV, \tag{4.54}$$

where $\overrightarrow{f^{vol}} = f_i^{vol}\vec{e_i}$ represent the bulk forces applied to a unit volume (for instance if bulk forces consist of gravity $\overrightarrow{f^{vol}} = -\rho g\vec{e_3}$), and the equations of motion are given by:

$$\rho \frac{D\vec{v}}{Dt} dV = d\vec{F}, \tag{4.55}$$

Hence, after dividing both sides by the volume element:

$$\boxed{\rho \left(\frac{\partial}{\partial t}\vec{v} + \left(v_i \frac{\partial}{\partial x_i} \right) \vec{v} \right) = \overrightarrow{f^{vol}} + \frac{\partial}{\partial x_j}\sigma_{ij}\vec{e_i}.} \tag{4.56}$$

To do computations using Eq. 4.56, we need to have a law relating the stress tensor σ to the behaviour of the fluid (a law that would play the role of Hooke's law in the case of fluids).

Example (perfect fluids and the Euler equation[1]). We will try several possibilities, but the simplest law one can propose is the isotropic law corresponding to surface forces that are purely normal:

$$\sigma_{ij}(\vec{x}, t) := -P(\vec{x}, t)\delta_{ij}, \tag{4.57}$$

[1]named after Leonhard Euler (1707-1783), who founded the discipline of fluid mechanics in 1755 by proposing this equation.

where P is a scalar function these are compression forces when P is positive. In this case, Eq. 4.56 becomes the Euler equation:

$$\rho\left(\frac{\partial}{\partial t}\vec{v}+\left(v_i\frac{\partial}{\partial x_i}\right)\vec{v}\right)=\overrightarrow{f^{vol}}-\frac{\partial P}{\partial x_i}\vec{e_i}. \tag{4.58}$$

It is the equation of motions of fluids that are completely characterized by their density ρ and pressure P. These fluids are called *perfect fluids*, or non-viscous fluids. To model the viscosity we will have to add extra terms to the stress tensor and to use the more general equation of motion, Eq. 4.56.

4.5 Viscous fluids

We want to go beyond the model of perfect fluids in order to describe the *fact* that some fluids resist flow more than others.

4.5.1 Boundary conditions on the velocity field

The no-slipping condition is imposed: on a fixed boundary, the velocity field is zero, more generally the velocity of the fluid *coincides with the velocity of the boundary* (see the flow between a fixed plaque and a moving plaque for an example). Intuitively, a viscous fluid *sticks to boundaries*[2].

4.5.2 Material law for Newtonian fluids, equations of motion

There is no *mathematical derivation* of a material law for fluids (there is very little theory when dealing with friction). All we can do is to propose mathematical forms for material laws translating some physical assumptions, and they will have to be confronted to exepriment. We will limit ourselves to the simplest class of material laws, in which friction is at most linear in the gradient of the velocity field (the study of fluids with more complex material laws is called *rheology*).

We already saw one possible material law for fluids, corresponding to an isotropic stress tensor (Eq. 4.57). It gives rise to Euler's equation when substituted into Eq. 4.41, and it corresponds to surface forces that are purely normal (from Cauchy's principle). Hence the material law 4.57 physically corresponds to neglecting friction forces in the fluid (in which case the fluid is called a *perfect fluid*).

We want to keep the isotropic term in the material law, but we have to include terms modeling friction. Newtonian fluids are fluids for which an extra term proportional to the first derivatives of the velocity field is included in the material law. The coefficient μ is called the viscosity of

[2] The word viscous comes from the Latin *viscosus*, which means *sticky*.

the fluid, and it only depends on the nature of the fluid:

$$\boxed{\sigma_{ij}(\vec{x},t) = -P(\vec{x},t)\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).} \tag{4.59}$$

This model corresponds to friction forces that are proportional to the relative velocity *layers of fluids*[3]. To substitute the material law (Eq. 4.59) into the equations of motion (Eq. 4.41), we need to compute the following derivatives for all i in $\{1,2,3\}$:

$$\frac{\partial \sigma_{ij}}{\partial x_j}(\vec{x},t) = -\frac{\partial P(\vec{x},t)}{\partial x_j}\delta_{ij} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial^2 v_j}{\partial x_j \partial x_i}\right). \tag{4.60}$$

For incompressible fluids, the divergence of the velocity field is zero (see Eq. 4.48), so the last term in the above equation vanishes (after permuting the two derivatives), hence the equations of motion become the Navier–Stokes equation[4]:

$$\boxed{\rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}.\vec{\nabla})\vec{v}\right) = \overrightarrow{f^{vol}} - \vec{\nabla}P + \mu\triangle\vec{v}.} \tag{4.61}$$

Remark. The notations $(\vec{v}.\vec{\nabla})$ and $\triangle$ for the differential operators are convenient to write the equation without indices (i.e. with no reference to a basis of $\mathbf{R}^3$), but for actual calculations it is crucial to bear in mind their actual meaning (derived in Section 2 and in Eq. 4.60 respectively. In Cartesian coordinates with a fixed basis $(\vec{e_1},\vec{e_2},\vec{e_3})$ we have

$$(\vec{v}.\vec{\nabla})\vec{v} = \left(v_i\frac{\partial}{\partial x_i}\right)(v_j\vec{e_j}) = \left(\left(v_i\frac{\partial}{\partial x_i}\right)v_j\right)\vec{e_j}, \tag{4.62}$$

$$\triangle\vec{v} = \left(\frac{\partial^2}{\partial x_i \partial x_i}\right)(v_j\vec{e_j}) = \frac{\partial^2 v_j}{\partial x_i \partial x_i}\vec{e_j}. \tag{4.63}$$

[3]For a justification of this claim, see the tutorials, where the velocity field of a viscous fluid between two parallel plaques is calculated

[4]named after Claude-Louis Navier (1785-1836), who worked as a civil engineer while putting elasticity and viscous fluid mechanics in a mathematically usable form and George Stokes (1819-1903), Lucasian profesor of physics at Cambridge who worked out the first few solutions of this equation, and interpreted them in terms of friction.

Tutorial Sheet 7: Planar Couette flow

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. Steady flow of a viscous fluid between two plaques.

A viscous fluid is flowing between two infinite plaques of equation $x_3 = 0$ and $x_3 = h$, where $\vec{e_3}$ is the ascending vertical (i.e. the gravity acceleration equals $-g\vec{e_3}$, where $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ is an orthonormal basis). The plaque of equation $x_3 = 0$ is fixed, the plaque of equation $x_3 = h$ is moving along the direction $\vec{e_1}$ with constant speed $V_0\vec{e_1}$. The fluid is incompressible, and its Euler velocity field is assumed to be stationary

$$\frac{\partial \vec{v}}{\partial t} = \vec{0}, \tag{4.64}$$

and to depend only on the x_1 and x_3 coordinates. We therefore write $\vec{v}$ as follows

$$\vec{v}(\vec{x}, t) = v(x_1, x_3)\vec{e_1}. \tag{4.65}$$

(i) Write the PDE expressing the conservation of mass in the fluid. Prove that v depends only on x_3.

(ii) The pressure P is also assumed to depend only on x_1 and x_3. The viscosity is denoted by ν, the density is denoted by ρ. Write the Navier–Stokes equations and justify carefully that they imply the following equations:

$$0 = -\frac{\partial P}{\partial x_1} + \nu \frac{\partial^2 v}{\partial x_3^2}, \tag{4.66}$$

$$0 = -\frac{\partial P}{\partial x_3} - \rho g, \tag{4.67}$$

(iii) What are the boundary conditions in the function v at $x_3 = 0$ and $x_3 = h$?

(iv) Integrate the equations 4.66 and 4.67, and conclude that:

$$P(x_1, x_3) = -\rho g x_3 + \text{constant}, \quad \text{and} \quad v(x_1, x_3) = V_0 \frac{x_3}{h}. \tag{4.68}$$

(iv) Calculate the force by which the fluid acts on the rectangle

$$\left\{ \vec{x} = x_i \vec{e_i} \in \mathbf{R}^3,\ 0 \le x_1 \le a,\ 0 \le x_2 \le b, x_3 = h \right\}, \tag{4.69}$$

which is part of the plaque at $x_3 = h$. Use this result to interpret the viscosity in terms of friction.

Lecture 8: the Couette flow

In this lecture we will solve the incompressible Navier–Stokes equation between to coaxial cylinders. The solution will allow us to suggest an experiment to measure the viscosity of a Newtonian fluid. The main technical difficulty comes from the cylindrical geometry and the non-uniform nature of the orthoradial vector.

4.6 The Couette flow

The Couette flow[5] is a steady flow of an incompressible fluid occupying the space between two coaxial cylinders of radii a and b, with $a < b$. The inner cylinder is fixed, while the external cylinder is rotating at constant speed ω radians per second. As a viscous fluid adheres to boundaries, the velocity field is zero on the inner cylinder, while it equals the velocity of the external cylinder on $r = b$.

Let us look for a velocity field that take the following form in cylindrical coordinates:

$$\vec{v}(r, \theta) = v(r)\vec{e_\theta}, \tag{4.70}$$

where v is a scalar function we will have to determine, which means that the fluid flows in the orthoradial direction, and the norm of the velocity respects the cylindrical symmetry (it depends only on the distance from the axis). We also assume that the length of the cylinder is large compared to a and b, so we did not include any dependence on z in the form (this is equivalent to considering an "infinite cylinder"). Moreover there is no explicit dependence in time in Eq. 4.6, which means that we are interested in a steady flow.

4.7 The cylindrical Couette flow

4.7.1 Cylindrical coordinates (see tutorial for detailed derivations)

For a scalar function of a point in $\mathbf{R}^3$ described by cylindrical coordinates $f : (r, \theta, z) \mapsto f(r, \theta, z)$, the gradient in cylindrical coordinates is expressed as

$$\boxed{\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{e_z}.} \tag{4.71}$$

[5]named after the French physicist Maurice Couette (1858-1943), who built a viscosimeter basisd on two coaxial cylinders.

We can compute the Laplacian of f in cylindrical coordinates by applying Stokes' theorem to the flux of $\vec{\nabla} f$ through the boundary of an elementary volume oriented towards the exterior. Consider a vector field $\vec{u} = u_r\vec{e_r} + u_\theta\vec{e_\theta} + u_z\vec{e_z}$ and apply Stokes' theorem to the flux of $\vec{u}$ through an elementary volume centered at point $\vec{x}$ and spanned by the three vectors $dr\vec{e_r}, rd\theta\vec{e_\theta}, dz\vec{e_z}$ (see tutorial for details):

$$dV = rd\theta drdz, \tag{4.72}$$

$$\begin{aligned} \mathrm{div}\vec{u}(r,\theta,z)dV &= (u_z(r,\theta,z+dz) - u_z(r,\theta,z))rd\theta dr \\ &+((r+dr)u_r(r+dr,\theta,z) - ru_r(r,\theta,z))d\theta dz \\ &+(u_\theta(r,\theta+d\theta,z) - u_\theta(r,\theta,z))drdz. \\ &= rdrd\theta dz\left(\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}\right), \end{aligned} \tag{4.73}$$

hence

$$\boxed{\mathrm{div}\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z}} \tag{4.74}$$

Substituting the gradient of f (Eq. 4.95) to $\vec{u}$, we obtain:

$$\boxed{\triangle f(r,\theta,z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) + \frac{\partial^2 f}{\partial z^2}.} \tag{4.75}$$

4.7.2 Rewriting the Navier–Stokes equations in cylindrical coordinates

The time derivative of the velocity field is zero, as usual for permanent flows:

$$\boxed{\frac{\partial\vec{v}}{\partial t} = \vec{0}.} \tag{4.76}$$

Consider the expression proposed in Eq. 4.6 for the velocity field in cylindrical coordinates. As usual we write down the differential operator used to compute the convection term, which from Eq. 4.95:

$$\vec{v}.\vec{\nabla} = \frac{v(r)}{r}\frac{\partial}{\partial\theta}. \tag{4.77}$$

We compute the convection term by acting with this differential on the velocity field, without forgetting that $\vec{e_\theta}$ is a not constant vector but depends on θ:

$$(\vec{v}.\vec{\nabla})\vec{v} = \frac{v(r)}{r}\frac{\partial}{\partial\theta}(v(r)\vec{e_\theta}) = \frac{v(r)^2}{r}\frac{\partial}{\partial\theta}\vec{e_r} = -\frac{v(r)^2}{r}\vec{e_r}. \tag{4.78}$$

Of course we obtain the same result if we use the basis $(\vec{e_1}, \vec{e_2}, \vec{e_3})$ to express the velocity field, because the vectors $\vec{e_1}$ and $\vec{e_2}$ are constant:

$$\boxed{(\vec{v}.\vec{\nabla})\vec{v} = \frac{v(r)}{r}\frac{\partial}{\partial\theta}(v(r)(-\sin\theta\vec{e_x} + \cos\theta\vec{e_y})) = \frac{v(r)^2}{r}(-\cos\theta\vec{e_x} - \sin\theta\vec{e_y}) = -\frac{v(r)^2}{r}\vec{e_r}.} \tag{4.79}$$

The pressure term is a simple application of the differential operator to the scalar function P:

$$\boxed{-\vec{\nabla}P = -\frac{\partial P}{\partial r}\vec{e_r} - \frac{1}{r}\frac{\partial P}{\partial \theta}\vec{e_\theta} - \frac{\partial P}{\partial z}\vec{e_z}.} \tag{4.80}$$

The Laplacian term can be computed in the same way, acting with the differential operator on the vector $\vec{v}$, not forgetting to differentiate the vectors of the basis:

$$\begin{aligned}\triangle\vec{v}(r,\theta) &= \tfrac{1}{r}\tfrac{\partial}{\partial r}\left(r\tfrac{\partial(v(r)\vec{e_\theta})}{\partial r}\right) + \tfrac{1}{r}\tfrac{\partial}{\partial \theta}\left(\tfrac{1}{r}\tfrac{\partial(v(r)\vec{e_\theta})}{\partial \theta}\right)\\ &= \left(\tfrac{1}{r}\tfrac{d}{dr}(rv'(r))\right)\vec{e_\theta} + \tfrac{1}{r}\tfrac{\partial}{\partial\theta}\left(-\tfrac{v(r)}{r}\vec{e_r}\right)\\ &= \left(\tfrac{1}{r}\tfrac{d}{dr}(rv'(r)) - \tfrac{v(r)}{r^2}\right)\vec{e_\theta}\end{aligned} \tag{4.81}$$

Collecting the scalar coefficients of the three vectors $\vec{e_r}, \vec{e_\theta}, \vec{e_z}$, we obtain the following system of equations:

$$\boxed{\begin{cases} -\rho\frac{v^2}{r} = -\frac{\partial P}{\partial r}\\ 0 = -\frac{1}{r}\frac{\partial P}{\partial \theta} + \frac{\mu}{r}\frac{d}{dr}(rv'(r)) - \mu\frac{v(r)}{r^2}\\ 0 = -\rho g - \frac{\partial P}{\partial z}\end{cases}} \tag{4.82}$$

4.7.3 Integration of the Navier–Stokes equations

The third equation of the system is the same as in Cartesian coordinates and just expresses the hydrostatic dependence of the pressure field. Integrating the second equation (the one corresponding to the coefficients along $\vec{e_\theta}$) is enough to determine the velocity field. First of all, we notice that $\partial P/\partial\theta$ depends only on r:

$$\frac{\partial P}{\partial \theta} = \mu\frac{d}{dr}(rv'(r)) - \mu\frac{v(r)}{r}, \tag{4.83}$$

hence there exists two functions of r only, call them C and D, such that

$$P(r,\theta,z) = C(r)\theta - \rho g z + D(r). \tag{4.84}$$

However, since θ is defined up to a multiple of 2π, the function $\theta \mapsto P(r,\theta,z)$ must be 2π-periodic for all values of r and z between the two cylinders. The only way to satisfy this periodicity condition is to have $C(r) = 0$. Hence the pressure does not depend on the orthoradial angle:

$$\frac{\partial P}{\partial \theta} = 0, \tag{4.85}$$

and Eq. 4.83 become an equation in v only:

$$\frac{d}{dr}(rv'(r)) - \frac{v(r)}{r} = 0. \tag{4.86}$$

We rewrite it as

$$v'' + \frac{v'}{r} - \frac{v}{r^2} = 0. \tag{4.87}$$

We integrate this equation once: there exists a constant D such that

$$v' + \frac{v}{r} = D. \tag{4.88}$$

The function $r \mapsto Dr/2$ is a particular solution of this equation, and we can introduce another constant E (and redefine the unknown constant D) such that

$$v(r) = \frac{E}{r} + Dr. \tag{4.89}$$

Since the radius of the inner cylinder is strictly positive, this expression is finite for all values of r corresponding to points between the two cylinders, and we can determine the two constants using the two boundary conditions:

$$v(a) = 0, \quad v(b) = b\omega. \tag{4.90}$$

Hence

$$\begin{cases} 0 & = \frac{E}{a} + Da \\ b\omega & = \frac{E}{b} + Db. \end{cases} \tag{4.91}$$

Hence

$$D = \frac{b^2\omega}{b^2 - a^2}, \quad E = -\frac{a^2 b^2 \omega}{b^2 - a^2}. \tag{4.92}$$

from which we obtain the expression of the velocity field:

$$\boxed{\vec{v}(r,\theta,z) = \left(\frac{b^2\omega}{b^2 - a^2}\left(-\frac{a^2}{r} + r\right)\right)\vec{e_\theta}.} \tag{4.93}$$

Tutorial Sheet 8: All things cylindrical (Couette acceleration and coordinates)

Exercise 0. Look for errors in the lecture notes and report your findings.

Exercise 1. Direct calculation of the acceleration in the Couette flow. Consider a particle of fluid between the two cylinders in the (steady, translation-invariant along the direction $\vec{e_z} = \vec{e_3}$) Couette flow, described by its position $r\vec{e_r}(\theta) + z\vec{e_z}$ in cylindrical coordinates, and a time t. Consider a very short interval of time ϵ, and the vector

$$\vec{U}(r, \theta, z, t, \epsilon) = \vec{v}(r\vec{e_r}(\theta) + z\vec{e_3} + \epsilon\vec{v}(r\vec{e_r}(\theta), t), t + \epsilon) - \vec{v}(r\vec{e_r}(\theta) + z\vec{e_3}, t),$$

where the velocity field is orthoradial and can be written in terms of an unknown scalar function v of the coordinate r only

$$\vec{v}(r\vec{e_r}(\theta) + z\vec{e_3}, t) = v(r)\vec{e_\theta}.$$

Notice that the trajectories of particles are circles, use this fact to expand $\vec{U}$ at first order in ϵ, work out the acceleration of the particle of fluid and check that the convection term is consistent with what was obtained by working out the convection term $(\vec{v}.\vec{\nabla})\vec{v}$ in cylindrical coordinates.

Exercise 2. Cylindrical coordinates, the gradient, divergence and Laplace operators. The purpose of this exercise is to derive the expression of some differential operators in cylindrical coordinates, from scratch.
(i) Consider the Cartesian coordinates (x_1, x_2, x_3) of any point M in $\mathbf{R}^3$ defined by the relation

$$\overrightarrow{OM} = x_i\vec{e_i}.$$

We define the cylindrical coordinates (r, θ, z) by

$$\overrightarrow{OM} = r\vec{e_r} + z\vec{e_3},$$

where $\vec{e_r}$ is a vector-valued function of θ, which can be made explicit by writing it as $\vec{e_r}(\theta) = \cos\theta\vec{e_1} + \sin\theta\vec{e_2}$, and $\vec{e_3} = \vec{e_z}$. Furthermore, define $\vec{e_\theta}(\theta)$ by $\vec{e_\theta}(\theta) = \frac{\partial\vec{e_r}}{\partial\theta}$. Put all these vectors and coordinates on a drawing, express r, $\cos\theta$ and $\sin\theta$ in Cartesian coordinates (i.e. in terms of x_1, x_2, x_3), and prove that $\frac{\partial\vec{e_\theta}}{\partial\theta} = -\vec{e_r}$.

(ii) Prove that $(\vec{e_r}, \vec{e_\theta}, \vec{e_3})$ is an orthonormal base.

Bear in mind that the vectors $\vec{e_r}$ and $\vec{e_\theta}$ depend on the point: they form a *local* orthonormal base.

(*iii*) The gradient $\vec{\nabla} f$ of a scalar function of n variables $f : \mathbf{R}^n \to \mathbf{R}$ (in our case $n = 3$) is defined by the relation

$$df = \vec{\nabla} f . d\vec{x} \tag{4.94}$$

Write the expression of $d\vec{x}$ in cylindrical coordinates, by differentiating the expression $\vec{OM} = r\vec{e_r} + x_3\vec{e_3}$ w.r.t. the variables (r, θ, z). Deduce the expression of the gradient in cylindrical coordinates:

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\vec{e_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e_\theta} + \frac{\partial f}{\partial z}\vec{e_z}. \tag{4.95}$$

(*iv*) Place on your drawing the three vectors you found in the expression of $d\vec{x}$ in a cylindrical coordinates. Consider the (infinitesimal) volume spanned by these three vectors, describe the normal vectors to the boundary of this volume. Express the infinitesimal volume in terms of $r, dr, d\theta, dz$.

(*v*) Write down the flux of a vector field $\vec{u} = u_r\vec{e_r} + u_\theta\vec{e_\theta} + u_z\vec{e_z}$ through this surface. Apply Stokes' theorem to prove that the divergence of the vector field is expressed as follows:

$$\mathrm{div}\vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z}. \tag{4.96}$$

(*vii*) Apply this formula to $\vec{u} = \vec{\nabla} f$ and conclude that the expression of the Laplacian operator is

$$\triangle f(r, \theta, z) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right) + \frac{\partial^2 f}{\partial z^2}. \tag{4.97}$$